Stats 117 Problem Set 9

Due: Monday, June 1 5:00 p.m. on Gradescope

Please show your work for each exercise. If you collaborate with someone else—this is fine—be sure to note that in your homework submission. You must each write up separate answer sets. Any starred exercise is optional: they are extra challenging theoretical exercises for further developing your mastery.

Question 9.1: Two companies, 1 and 2, are founded at a fixed time (we'll call it 0). There are adverse events that affect the individual companies, where if adverse event a_1 occurs, then company 1 collapses; if adverse event a_2 occurs, then company 2 collapses. Each company continues to exist until either there is a stock market crash or the associated adverse event a_1 or a_2 occurs.

Assume that the time X_1 until adverse event a_1 is $X_1 \sim \mathsf{Exp}(\lambda_1)$, the time X_2 until adverse event a_2 is $X_2 \sim \mathsf{Exp}(\lambda_2)$, and the time until a stock market crash (very bad) is $V \sim \mathsf{Exp}(\lambda_0)$. Each of X_1, X_2, and V is independent.

(a) Let T_1 be the time the first company collapses. Give its distribution.

(b) Let T_2 be the time the second company collapses. Give its distribution.

(c) For any fixed $t_1, t_2 \geq 0$, give $P(T_1 \geq t_1, T_2 \geq t_2)$.

(d) Use the results of the previous parts to give the joint cumulative distribution function (CDF) of the pair (T_1, T_2) of survival times, that is, $F_{T_1,T_2}(t_1, t_2) = P(T_1 \leq t_1, T_2 \leq t_2)$.

Question 9.2: By the Biot-Savart law, a point X meters from a wire carrying a current I experiences magnetic force $H = 2I/X$. Suppose the distance X of the point is uniform over $[3, 5]$, and that the current I is independent of X and uniform on $[10, 20]$. Give $\mathbb{E}[H]$.

Question 9.3: The *risk* of an investment allocation is its variance. Suppose we consider investing in 3 assets, where the return of asset i after a certain time period is X_i. We invest a proportion of our wealth, denoted w_i, in investment X_i, so that our total realized payout is $w_1X_1 + w_2X_2 + w_3X_3$. We call the *risk adjusted return* of an investment allocation

$$R(w) := \mathbb{E}[w_1X_1 + w_2X_2 + w_3X_3] - \lambda \cdot \mathrm{Var}(w_1X_1 + w_2X_2 + w_3X_3),$$

where $\lambda \geq 0$ is the amount we penalize the risk of the investment strategy.

(a) Let $\mu_i = \mathbb{E}[X_i]$ be the expected return of asset i and $C_{ij} = \mathrm{Cov}(X_i, X_j)$ the covariance of the returns from assets i and j. Using properties of expectation and covariance, give a formula for $R(w)$ depending only on w_i, μ_i, and the covariance values C_{ij} for $i, j \in \{1, 2, 3\}$.

(b) Suppose that asset 1 corresponds to investing in the bank, so $X_1 = 1.05$ with probability one. Give C_{1j} and C_{j1} for each j.

(c) Let $W, Z \sim \mathsf{N}(0, 1)$ be independent and standard normal. Let $X_2 = 1.2 + W$ and $X_3 = 1.5 + W + Z$. Give $\mathrm{Var}(X_2) = \mathrm{Cov}(X_2, X_2)$, $\mathrm{Var}(X_3) = \mathrm{Cov}(X_3, X_3)$, and $\mathrm{Cov}(X_2, X_3)$.

(d) If $\lambda = .3$, give the risk-adjusted return $R(w)$ for the distributions of returns above when $w_1 = w_2 = w_3 = \frac{1}{3}$, an equal allocation of resources. Is this better or worse than the allocation $w_1 = 1$ and $w_2 = w_3 = 0$, which puts all the money in the bank?

Question 9.4: Let the points (X, Y) be uniformly distributed within the circle of radius r, that is, they have density

$$f(x,y) = \begin{cases} c & \text{if } x^2 + y^2 \le r^2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Give c (that is, the value of c such that $\iint f(x,y)dxdy = 1$).

(b) Give the marginal density of Y.

(c) Is Y uniform? If not, at what values of Y is its density $f_Y(y)$ maximized?

(d) Give the conditional density $f_{X|Y}(x \mid y)$ of X given $Y = y$.

Question 9.5: Let a particle begin at location zero on an infinite line. It undergoes diffusion in one dimension if its position X_t at time t follows a $\mathsf{N}(0,t)$ distribution.

(a) Give the expected distance $\mathbb{E}[|X_t|]$ of the particle from the orgin at time t (where $t \ge 0$).

The particle undergoes a 2-dimensional diffusion if its coordinates $(X_t, Y_t) \in \mathbb{R}^2$ at time t independently follow $\mathsf{N}(0,t)$ distributions. (Note that $(X_0, Y_0) = (0,0)$.)

(b) Give $\mathbb{E}[\sqrt{X_t^2 + Y_t^2}]$, the expected distance of the particle from the origin at time $t \ge 0$. *Hint.* Integrate in "solids of revolution" as in Example 23.9. Your result for this answer should have the same power of t as part (a).

Question 9.6* (The existence of a Poisson distribution)**:** In this question, you will *derive* the Poisson distribution from exponential distributions. (We take it on faith, based on radioactive decay, that the exponential distribution exists.)

(a) Use the convolution formula (Proposition 26.2) to show that the density of $Z = X + Y$, where $X \sim \mathsf{Exp}(\lambda)$ and $Y \sim \mathsf{Exp}(\lambda)$, is

$$f(z) = \lambda^2 z e^{-\lambda z} \quad \text{for } z \ge 0.$$

(b) Use induction to show that if $X_1, X_2, \ldots, X_k$ are independent $\mathsf{Exp}(\lambda)$ random variables, then $Z_k := X_1 + X_2 + \cdots + X_k$ has density

$$f_k(z) = \frac{\lambda^k z^{k-1}}{(k-1)!} e^{-\lambda z}.$$

(c) Let $X_i \sim \mathsf{Exp}(\lambda)$ indicate independent times between a set of events, so $Z_k = X_1 + \cdots + X_k$ is the time of the kth event. Let $T > 0$. Then we claim that the number of events occuring in the time window $[0,T]$ is a $\mathsf{Poisson}(\lambda T)$ random variable. Prove this. That is, show

$$P(Z_k \le T, Z_k + X_{k+1} > T) = \frac{(\lambda T)^k}{k!} e^{-\lambda T}.$$

(d) Why does this imply that $N_T = \max\{k : X_1 + \cdots + X_k \le T\}$, the number of events before time T, satisfies $N_T \sim \mathsf{Poisson}(\lambda T)$? *Hint.* For continuous random variables X and Y,

$$\begin{aligned} P(X \in A, Y \in B) &= \int_B \int_A f_{X,Y}(x,y)dxdy = \int_B \int_A f_{X|Y}(x \mid y) f_Y(y) dxdy \\ &= \int_B P(X \in A \mid Y = y) f_Y(y) dy. \end{aligned}$$