

Stats 117 Lecture 1

The Classical Definition of Probability

Prof. John Duchi

Autumn 2026

1-1

Organization

- ▶ why we need a mathematical framework
- ▶ sample spaces and events
- ▶ the classical definition of probability
- ▶ applications
 - ▶ cards, genetics, dice
 - ▶ Secret Santa, Monty Hall
- ▶ what the classical definition *cannot* do

1-2

Motivation: *People v. Collins* (1968)

- ▶ a purse is snatched in Los Angeles; witnesses describe
 - ▶ a woman with blond hair in a ponytail
 - ▶ fleeing in a yellow car
 - ▶ driven by a Black man with a beard and mustache
- ▶ police find a couple matching the description
- ▶ prosecutor multiplies “probabilities” of each characteristic:

$$\frac{1}{10} \cdot \frac{1}{4} \cdot \frac{1}{10} \cdot \frac{1}{3} \cdot \frac{1}{10} \cdot \frac{1}{1000} = \frac{1}{12,000,000}$$

- ▶ “surely this couple is guilty”

1-3

What went wrong?

- ▶ **question the prosecutor answered:** what is the chance a *randomly chosen* couple matches the description?
- ▶ **question that matters:** given that *some* matching couple committed the crime, what is the chance *this* couple did?
- ▶ **defense:** chance that *at least two* matching couples exist in Los Angeles is over 40%

1-4

More motivation

- ▶ Finance: what is $\Pr(\text{lose money})$?
- ▶ Epidemiology: model spread of disease
- ▶ Insurance: price policies
- ▶ Science: what is probability effect is real?

1-5

Even the experts get this wrong

three errors by very smart people:

- ▶ **Cardano** (~ 1560): roll a die 3 times. Chance of at least one six is

$$3 \cdot \frac{1}{6} = \frac{1}{2}.$$

- ▶ **Leibniz** (~ 1700): rolling two dice, an 11 and a 12 are equally likely, since each can be rolled in only one way.
- ▶ **d'Alembert** (1754): toss a coin twice. The outcomes are H , TH , TT , so

$$P(\text{at least one heads}) = \frac{2}{3}.$$

1-6

Sample spaces

Definition (Sample space)

The *sample space* of an experiment, denoted Ω , is the set of all possible outcomes of the experiment.

examples:

- ▶ spin a roulette wheel:
- ▶ toss a coin twice:
- ▶ roll two dice:

1-7

Events

Definition (Event)

An *event* A is a subset of the sample space Ω , i.e., a set of outcomes.

- ▶ an event *happens* if the outcome of the experiment lies in A
- ▶ English descriptions are events:

$$\{\text{at least one heads}\} = \{HH, HT, TH\} \subset \Omega$$

1-8

The classical definition of probability

Definition (Classical definition)

If the sample space Ω consists of *equally likely* outcomes, then the probability of an event A is

$$P(A) = \frac{|A|}{|\Omega|},$$

where $|S|$ is the cardinality (number of elements) of a set S .

- ▶ two things to do: (i) describe Ω so outcomes are equally likely, (ii) count
- ▶ note $0 \leq P(A) \leq 1$, $P(\Omega) = 1$, $P(\emptyset) = 0$

1-9

Roulette

- ▶ American roulette wheel has 38 pockets: 1, 2, ..., 36 plus 0 and 00
- ▶ 18 are red, 18 are black, 2 are green
- ▶ wheel is balanced, so all 38 pockets equally likely

so

$$P(\text{land on red}) = \frac{18}{38} = \frac{9}{19} \approx .4737$$

1-10

Cards

standard deck: 52 cards, 4 suits $\times$ 13 ranks

► $P(\text{red card}) =$

► $P(\text{queen}) =$

► $P(\text{Three of Clubs}) =$

1-11

Genetics: albinism

- albinism is recessive: a child is albino only with genotype aa
- both parents are *carriers*, genotype Aa
- child gets one allele from each parent, equally likely

		father	
		A	a
mother	A	AA	Aa
	a	Aa	aa

$$\Omega = \{AA, Aa, aA, aa\}, \quad P(\text{child is albino}) = \frac{1}{4}$$

1-12

Dice: the come-out roll in craps

shooter rolls two dice: **wins** on 7 or 11, **loses** on 2, 3, or 12, else that number is “the point”

first die	second die					
	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

$$P(\text{win}) = \frac{8}{36} \approx .2222, \quad P(\text{lose}) = \frac{4}{36} \approx .1111$$

1-13

Secret Santa

- ▶ four friends each draw a name from a hat
- ▶ a draw is *good* if nobody draws their own name
- ▶ what is $P(\text{good draw})$?

- ▶ Ω = all ways to assign names =

- ▶ A =

1-14

Secret Santa: counting

write outcome as (a_1, a_2, a_3, a_4) , where a_i is the name person i draws. Then $|\Omega| = 4! = 24$, and the good outcomes are

2143	2341	2413
3142	3412	3421
4123	4312	4321

so

$$P(\text{no one draws their own name}) = \frac{9}{24} = .375$$

1-15

The Monty Hall problem

- ▶ three doors: one hides a car, two hide goats
- ▶ you pick a door (say door 1)
- ▶ Monty, who knows where the car is, opens one of the other doors to reveal a goat
- ▶ he offers you the chance to switch

should you switch?

1-16

Monty Hall: sample space

let the outcome be the door hiding the car; you always pick door 1

car behind	switching wins?	staying wins?
door 1		
door 2		
door 3		

1-17

Limitations of the classical definition

$P(A) = |A|/|\Omega|$ requires equally likely outcomes. It fails when

- ▶ outcomes are *not* equally likely
 - ▶ a bent coin, a loaded die, a person's genotype in a population
- ▶ the sample space is *infinite*
 - ▶ how many tosses until the first heads?
 - ▶ how long until a lightbulb burns out?

and even when outcomes *are* equally likely, Ω may be far too big to list

next time: techniques for counting, so we never have to list Ω again

1-18