

Stats 117 Lecture 2

Techniques for Counting

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Organization

- ▶ the fundamental principle of counting
- ▶ four ways of counting
 - ▶ does order matter?
 - ▶ with or without replacement?
- ▶ applications: dice, birthdays, poker
- ▶ combinatorial identities

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Recapitulation

- ▶ sample space Ω is the set of possible outcomes, event $A \subset \Omega$ is a set of outcomes
- ▶ if outcomes in Ω *equally likely*,

$$P(A) = \frac{|A|}{|\Omega|}$$

so everything reduces to counting.

- ▶ last time: list all outcomes (24 Secret Santa draws, 36 dice rolls)
- ▶ today: counting *without* listing

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The fundamental principle of counting

Theorem (Fundamental principle of counting)

Suppose we perform n actions in sequence, and there are k_i ways to perform the i th action, no matter how the previous actions were performed. Then the number of ways to perform all n actions is

$$\prod_{i=1}^n k_i = k_1 \times k_2 \times \cdots \times k_n.$$

- ▶ “no matter how the previous actions were performed”

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Example: dressing

I own 3 pairs of shoes, 2 pairs of shorts, and 4 shirts, of which 2 are long-sleeved. I grab one of each at random.

► $|\Omega| =$

► $P(\text{long-sleeved shirt}) =$

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Four ways of counting

we choose k items from n .

► does order matter?

- ice cream *cone*: chocolate on vanilla $\neq$ vanilla on chocolate
- ice cream *cup*: scoops just there

► with or without replacement?

	order matters (cone)	order does not (cup)
without replacement		
with replacement		

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Order matters, with replacement

Proposition

The number of ways to select k items from n items, where order matters and selection is with replacement, is

$$n^k.$$

examples:

► roll a die n times:

► toss a coin n times:

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Cardano's problem, resolved

roll a fair die n times. What is $P(\text{at least one six})$?

► easier to count the *complement*: no sixes at all

$$P(\text{no sixes}) =$$

$n =$	3	4	5	10
$P(\text{at least one six})$	.4213	.5177	.5981	.8385

Cardano wanted $n = 3...$

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Order matters, without replacement

Proposition

The number of ways to select k items from n items, where order matters and selection is without replacement, is

$$n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}.$$

Corollary

number of ways to arrange n distinct objects (a permutation) is $n!$.

example: how many ways can 10 horses finish win, place, and show?

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The birthday problem

k people are in a room. What is the chance two share a birthday?

- ▶ assume 365 days, all equally likely, birthdays unrelated
- ▶ $|\Omega| =$

- ▶ count the complement: all k birthdays *distinct*

- ▶ so

$$P(\text{all distinct}) =$$

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The birthday problem: the numbers

k	10	23	30	40	50	60
$P(\text{some match})$	.117	.507	.706	.891	.970	.994

- ▶ 23 people suffice for better-than-even chance
- ▶ why so few? there are $\binom{23}{2} = 253$ *pairs* of people

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Order does not matter, without replacement

Proposition

The number of ways to select k items from n items, where order does not matter and selection is without replacement, is

$$\binom{n}{k} := \frac{n!}{k! (n - k)!},$$

read “ n choose k .”

why: count ordered selections, then divide out the orderings

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Poker: a full house

deal 5 cards from a 52-card deck. *Full house* is three cards of one rank and two of another.

- ▶ Ω = deals of five cards, so $|\Omega| =$
- ▶ build a full house by four actions:

- ▶ so

$$|A| =$$

$$\text{and } P(\text{full house}) = 3744/2,598,960 \approx .0014$$

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Exactly half heads

toss a fair coin $2k$ times. What is $P(\text{exactly } k \text{ heads})$?

- ▶ outcomes $|\Omega| = 2^{2k}$
- ▶ k heads: choose which k of $2k$ positions

$$P(\text{exactly } k \text{ heads}) =$$

$2k$	2	10	20	100	1000
$P(\# \text{ H} = k)$	.500	.246	.176	.0796	.0252

moral: k most likely number of heads, but probability $\rightarrow 0$

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Order does not matter, with replacement

3 scoops in cup, chosen from 5 flavors, repeats allowed. Represent a choice by *stars and bars*:

★★ || ★ || two scoops of flavor 1, one of flavor 3

|| | ★ ★ ★ | three scoops of flavor 4

★ | ★ || | ★ one each of flavors 1, 2, and 5

- ▶ k stars and $n - 1$ bars, row of $k + n - 1$ symbols
- ▶ choice determined by where stars go

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Stars and bars

Proposition

The number of ways to select k items from n items, where order does not matter and selection is with replacement, is

$$\binom{k + n - 1}{k} = \binom{k + n - 1}{n - 1}.$$

ice cream: $\binom{3+5-1}{3} = \binom{7}{3} = 35$ cups

warning: these 35 cups are not equally likely if you pick scoops at random

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Stars and bars: quantum physics

energy quanta system has m modes (harmonic oscillator degrees of freedom), total energy k

- ▶ mode i has n_i energy quanta

$$n_1 + \cdots + n_m = k$$

- ▶ how many energy allocations?

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Summary: four ways of counting

choosing k items from n :

	order matters (cone)	order does not (cup)
without replacement		
with replacement		

- ▶ three of these four are used constantly; fourth is harder

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Combinatorial identities: choosing a captain

Proposition

For $1 \leq k \leq n$,

$$n \binom{n-1}{k-1} = k \binom{n}{k}.$$

proof: count teams of k players with a captain in two ways

► captain first, then teammates:

► team first, then captain:

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Pascal's identity

Proposition

For $1 \leq k \leq n-1$,

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

proof: count teams of k from n people, splitting on whether Alice is on the team.

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