

Stats 117 Lecture 3

Axiomatic Definition of Probability

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Autumn 2026

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Organization

- ▶ failures of classical definition
- ▶ events: union, intersection, complement
- ▶ Kolmogorov axioms
- ▶ examples
- ▶ meaning of probability

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Recap

- ▶ outcomes in Ω equally likely: $P(A) = |A|/|\Omega|$
- ▶ lecture 2: compute $|A|$ and $|\Omega|$ without listing

today: two experiments where naive definition fails

- ▶ toss a coin until it lands heads
- ▶ throw a dart at a dartboard

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Breakdown 1: tossing until heads

toss fair coin until heads. Give

$$P(\text{first heads on toss } 2)$$

- ▶ infinite $\Omega = \{H, TH, TTH, TTTH, \dots\}$
- ▶ $A = \{TH\}$
- ▶ classical definition:

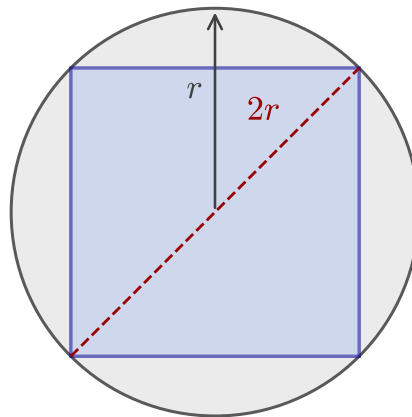
$$P(A) = \frac{|A|}{|\Omega|} = \frac{1}{\infty} = 0?$$

worse: $P(\text{first heads on an even toss})$ gives ∞/∞

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Breakdown 2: throwing darts

dart lands uniformly on a circular board of radius r



- ▶ $|\Omega|$ uncountably infinite
- ▶ answer should be $P(\text{square}) = 2/\pi \approx .6366$

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What should we do?

- ▶ don't just derive probabilities by counting
- ▶ Kolmogorov (1933): say what any definition of "probability" must satisfy
- ▶ check that counting and area both satisfy *axioms* (these new rules)

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Events as sets

events $A, B \subset \Omega$. Then

- ▶ **intersection** $A \cap B$: both A and B happen
- ▶ **union** $A \cup B$: A or B (or both) happens
- ▶ **complement** A^c : A does not happen
- ▶ **mutually exclusive** (disjoint): $A \cap B = \emptyset$

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Roulette: red and even

bet on red (A) and on even (B). $|\Omega| = 38$, $|A| = |B| = 18$.

- ▶ $P(\text{win both}) = P(A \cap B) =$

- ▶ $P(\text{win at least one}) = P(A \cup B) =$

careful: $|A \cup B| \neq |A| + |B|$: reds and evens overlap

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De Morgan's laws

Theorem (De Morgan)

$$(A \cup B)^c = A^c \cap B^c \quad \text{and} \quad (A \cap B)^c = A^c \cup B^c.$$

“not (red or even)” = “not red *and* not even”

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Kolmogorov's axioms

Definition (Axiomatic definition of probability)

A *probability function* P assigns a value to each event $A \subset \Omega$ with

1. (non-negativity) $P(A) \geq 0$ for every event A
2. (normalization) $P(\Omega) = 1$
3. (countable additivity) if $A_1, A_2, \dots$ are disjoint, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

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Counting is a probability function

let Ω be finite and set $P(A) = |A|/|\Omega|$. Check:

▶ non-negativity:

▶ normalization:

▶ additivity:

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Loaded die

nothing forces the outcomes to be equally likely:

outcome	1	2	3	4	5	6
P	.1	.1	.1	.1	.1	.5

▶ all three axioms hold

▶ $P(\text{roll is even}) =$

moral: axioms define rules, don't make one correct

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Fixing infinite coin flips (Breakdown 1)

toss a fair coin forever; $A_n = \{\text{first heads on toss } n\}$. Assign

$$P(A_n) = \frac{1}{2^n}.$$

► legal? need $\sum_{n=1}^{\infty} P(A_n) = 1$:

► $P(\text{first heads on toss } 2) =$

► $P(\text{first heads on an even toss}) =$

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Fixing the dart board (Breakdown 2)

dart lands uniformly on a board of radius r . Assign

$$P(\text{dart lands in } A) = \frac{\text{area of } A}{\text{area of dartboard}}.$$

► legal?

► inscribed square:

$$P(\text{land in square}) =$$

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What does a probability *mean*?*

axioms say how probabilities behave, but give no meaning

- ▶ **frequentist**: long-run fraction of repetitions,

$$P(A) = \lim_{n \rightarrow \infty} \frac{\#\{\text{first } n \text{ trials where } A \text{ happens}\}}{n}$$

- ▶ **subjectivist**: degree of “belief”

a roulette wheel can be spun forever.

- ▶ will it rain in Palo Alto tomorrow?
- ▶ who will win the election?

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Why a (gambling) subjectivist obeys the axioms

- ▶ if they don't I will clean them out

Football game (49ers vs. Cardinals): Pat believes

$$P(49ers \text{ by } \geq 5) = .40, \quad P(Cardinals \text{ by } \geq 5) = .20, \\ P(\text{spread is at least } 5) = .70.$$

- ▶ events mutually exclusive, so violates additivity

gambling game: buy/sell tickets with \$1 payout

- ▶ belief of .7 means willingness to pay \$.70 for ticket
- ▶ belief of .4 means willingness to sell ticket for \$.40

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A Dutch book

Pat's cash today: $-.70 + .40 + .20 = -\$.10$. On Sunday:

	49ers by ≥ 5	Cardinals by ≥ 5	neither
"either" ticket			
"49ers" ticket			
"Cardinals" ticket			
Pat's total			

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A warning*

axioms also rule things out

- ▶ can we pick a natural number "uniformly at random"?
- ▶ i.e. is there P with $P(\{n\}) = c$ for every $n \in \mathbb{N}$?

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