

Stats 117 Lecture 4

Properties of Probability Functions

Prof. John Duchi

Autumn 2026

4-1

Organization

- ▶ finite (disjoint) unions
- ▶ complement rule
- ▶ subset rule
- ▶ more unions
 - ▶ union bound
 - ▶ inclusion-exclusion

4-2

Recap

probability function P satisfies

1. $P(A) \geq 0$
2. $P(\Omega) = 1$
3. $A_1, A_2, \dots$ disjoint implies $P(\bigcup_i A_i) = \sum_i P(A_i)$

today: everything else follows from these three

- only have Axiom 3: break events into disjoint pieces

4-3

Finite sums

Proposition

For mutually exclusive events $A_1, A_2, \dots, A_n$,

$$P(A_1 \cup A_2 \cup \dots \cup A_n) = P(A_1) + \dots + P(A_n)$$

4-4

Complement rule

Proposition

For any event A ,

$$P(A^c) = 1 - P(A).$$

4-5

$$0 \leq P(A) \leq 1$$

Proposition

For any event A , $0 \leq P(A) \leq 1$.

proof:

note: axioms gave $P(A) \geq 0$; must prove upper bound

4-6

Cardano's problem, finished

roll a die n times; how large must n be so that $P(\text{at least one six}) \geq \frac{1}{2}$?

► complement rule:

$$P(\text{at least one six}) =$$

► solve $1 - (5/6)^n \geq \frac{1}{2}$:

4-7

Subset rule

Proposition

If $B \subset A$, then $P(B) \leq P(A)$.

4-8

The Linda problem (1970s)

Linda is 31, single, outspoken, and very bright. She majored in philosophy. As a student she was deeply concerned with discrimination and social justice, and participated in anti-nuclear demonstrations.

which is more probable?

- (a) Linda is a bank teller
- (b) Linda is a bank teller and is active in the feminist movement

4-9

All heads, forever

toss a fair coin forever; $A_i = \{\text{toss } i \text{ is heads}\}$.

$$P(\text{every toss is heads}) = ?$$

- ▶ for every n , $\bigcap_{i=1}^{\infty} A_i \subset \bigcap_{i=1}^n A_i$
- ▶ by the subset rule

$$P\left(\bigcap_{i=1}^{\infty} A_i\right) \leq$$

- ▶ true for every n , so let $n \rightarrow \infty$

4-10

Inclusion/exclusion (addition rule)

Proposition

For any events A and B ,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

proof: disjoint sum

4-11

Roulette, again

$A = \{\text{red}\}$, $B = \{\text{even}\}$, with

$$P(A) = P(B) = \frac{18}{38}, \quad P(A \cap B) = \frac{8}{38}$$

► $P(\text{win at least one bet}) =$

sanity check: without the correction we would get $36/38$

4-12

Union bound

Proposition

For any events $A_1, A_2, \dots$,

$$P\left(\bigcup_i A_i\right) \leq \sum_i P(A_i).$$

- ▶ always holds
- ▶ equality when the A_i are disjoint
- ▶ cheap, usually all you need for “nothing goes wrong”

4-13

Union bound: airplane safety

airlines ask for “ m 9s” of reliability

$$P(\text{no system fails}) \geq \underbrace{.99 \dots 9}_{m \text{ 9s}}$$

- ▶ $F_i = \{\text{system } i \text{ fails}\}$; we need $P\left(\bigcup_{i=1}^{35} F_i\right) \leq 10^{-m}$
- ▶ union bound:

$$P\left(\bigcup_{i=1}^{35} F_i\right) \leq$$

- ▶ suffices that $p \leq$

4-14

When the union bound is useless

Secret Santa, four friends; $E_i = \{\text{person } i \text{ draws own name}\}$.

► $P(E_i) = \frac{3!}{4!} = \frac{1}{4}$

► union bound:

$$P\left(\bigcup_{i=1}^4 E_i\right) \leq \sum_{i=1}^4 \frac{1}{4} = 1$$

useless: we already knew $P \leq 1$

the E_i overlap a lot, so the bound double-counts badly. We need an *exact* formula.

4-15

Inclusion-exclusion: three events

Theorem

$$\begin{aligned} P(A \cup B \cup C) = & P(A) + P(B) + P(C) \\ & - P(A \cap B) - P(A \cap C) - P(B \cap C) \\ & + P(A \cap B \cap C). \end{aligned}$$

why:

4-16

Inclusion-exclusion: n events

Theorem

$$\begin{aligned} P\left(\bigcup_{i=1}^n A_i\right) &= \sum_i P(A_i) - \sum_{i < j} P(A_i \cap A_j) \\ &\quad + \sum_{i < j < k} P(A_i \cap A_j \cap A_k) - \cdots \\ &\quad + (-1)^{n+1} P\left(\bigcap_{i=1}^n A_i\right). \end{aligned}$$

- ▶ add singles, subtract pairs, add triples, ...
- ▶ exact, but $2^n - 1$ terms
- ▶ proof is super tedious for now

4-17

Secret Santa, exactly

by symmetry, an intersection of k of the E_i has probability $(4-k)!/4!$, and there are $\binom{4}{k}$ of them:

$$P\left(\bigcup_{i=1}^4 E_i\right) =$$

- ▶ so $P(\text{no one draws own name}) = 1 - .625 = .375$
- ▶ agrees with $9/24$ (counted by hand in Ch. 1)

4-18

Which tool when?

- ▶ “at least one” $\Rightarrow$ try the **complement** first
- ▶ two events $\Rightarrow$ **addition rule**
- ▶ many “rare” events or complex dependencies $\Rightarrow$ **union bound**
- ▶ many events, need exact $\Rightarrow$ **inclusion-exclusion**