

Stats 117 Lecture 5

Conditional Probability

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Organization

- ▶ definition of conditional probability
 - ▶ the boy-girl paradox
- ▶ multiplication rule
 - ▶ many events
- ▶ conditional probabilities are probabilities
- ▶ the gambler's fallacy

5-2

Is the wheel due for black?

roulette wheel lands on red 9 spins in a row. Should you bet on black?

► $P(\text{all 10 spins red}) = \frac{18^{10}}{38^{10}} \approx .00057$

► $P(1 \text{ black, 9 red in 10 spins}) = \frac{\binom{10}{1} 18^9}{38^{10}} \approx .0057$

so black is 10 times as likely?

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Secret Santa, with information

four friends draw names. Friend 1: *"I did not draw my own."*

1234	2134	3124	4123
1243	2143	3142	4132
1324	2314	3214	4213
1342	2341	3241	4231
1423	2413	3412	4312
1432	2431	3421	4321

► before: $P(\text{friend 2 draws own name}) =$

► after: $P(\text{friend 2 draws own name given friend 1 doesn't}) =$

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Restricting the sample space

knowing B happened, only outcomes in B possible

idea: probability of A given B is the fraction of B that A takes

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Conditional probability

Definition

For events A and B with $P(B) > 0$, the *conditional probability of A given B* is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

- ▶ read “probability of A given B ”
- ▶ undefined if $P(B) = 0$

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Secret Santa, formally

$$A_i = \{\text{friend } i \text{ draws own name}\}$$

$$P(A_2 \mid A_1^c) =$$

intuition: if friend 1 did not draw own name, more likely to have drawn friend 2's

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The boy-girl paradox

family has two children; $\Omega = \{BB, BG, GB, GG\}$ (older first)

Scenario 1: "One of our two children is on the women's swim team."

$$P(2 \text{ girls} \mid \geq 1 \text{ girl}) =$$

Scenario 2: "Our older child is on the women's swim team."

$$P(2 \text{ girls} \mid \text{older is girl}) =$$

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The boy-girl paradox: why different?

- ▶ scenario 1 leaves ways to also have a boy
- ▶ scenario 2 leaves way to also have a boy

lesson: do not trust intuition, apply the definition carefully

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The boy-girl paradox: scenario 3

“One of our children is walking over now. She is on the women’s swim team.”

- ▶ meet a *random* child: need a bigger sample space

$$\Omega = \{\underline{B}B, \underline{B}G, \underline{G}B, \underline{G}G, B\underline{B}, B\underline{G}, G\underline{B}, G\underline{G}\}$$

(underline = child we meet)

$$P(2 \text{ girls} \mid \text{meet girl}) =$$

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Two spades

draw two cards from a shuffled deck. $P(\text{both spades})$?

- ▶ first card spade: of the time
- ▶ given that, second card spade: of the time
- ▶ both:

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Multiplication rule

Proposition

If $P(A) > 0$ and $P(B) > 0$, then

$$P(A \cap B) = P(B)P(A | B) = P(A)P(B | A).$$

proof:

- ▶ useful when $P(A | B)$ is easy but $P(A \cap B)$ is hard

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1992 NBA draft lottery

11 teams; worst record gets 11 balls, . . . , best gets 1 (66 total).
Draw a ball, remove all that team's balls, draw again.

Rockets	1	Kings	6
Hawks	2	Bullets	7
76ers	3	Nuggets	8
Hornets	4	Mavericks	9
Bucks	5	Magic	10
		Timberwolves	11

Magic (Shaq) 1st, Hornets (Mourning) 2nd:

$$P(\text{Magic 1st, Hornets 2nd}) =$$

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Is the second card a spade?

► $P(S_2 \mid S_1) =$

► $P(S_2 \mid S_1^c) =$

► two disjoint ways to have S_2 :

$$P(S_2) = P(S_1 \cap S_2) + P(S_1^c \cap S_2) =$$

symmetry: every position in the deck looks the same

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Multiplication rule: many events

notation: $P(A_1, \dots, A_n) = P(A_1 \cap \dots \cap A_n)$; comma = “and”

Theorem

For events $A_1, \dots, A_n$ with positive probability,

$$P(A_1, \dots, A_n) = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1, A_2) \\ \dots P(A_n \mid A_1, \dots, A_{n-1}).$$

why:

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Order does not matter

any ordering of the events works, e.g.

$$P(A_1, A_2, A_3) = P(A_3)P(A_2 \mid A_3)P(A_1 \mid A_2, A_3) \\ = P(A_2)P(A_1 \mid A_2)P(A_3 \mid A_1, A_2)$$

- ▶ $n!$ theorems for the price of one
- ▶ choose the order where conditionals are easy

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Probability of a flush

five-card hand; flush = all five cards the same suit

► four disjoint suits:

$$P(\text{flush}) = 4 P(\spadesuit \text{ flush})$$

► multiplication rule with $S_i = \{\text{card } i \text{ spade}\}$:

$$P(\spadesuit \text{ flush}) =$$

so $P(\text{flush}) \approx$

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Conditional probabilities are probabilities

Proposition

If $P(B) > 0$, then $\tilde{P}(A) = P(A \mid B)$ is a probability function.

check the axioms:

1. $\tilde{P}(A) \geq 0$:

2. $\tilde{P}(\Omega) = 1$:

3. $A_1, A_2, \dots$ disjoint:

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Consequences

every rule from last time holds for $P(\cdot \mid B)$:

▶ $P(A^c \mid B) =$

▶ $0 \leq P(A \mid B) \leq 1$

▶ $A_1 \subset A_2$ implies

▶ $P(A_1 \cup A_2 \mid B) =$

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Linda, revisited

multiplication rule:

$$P(\text{teller and feminist}) =$$

new information: all of Linda's close friends are feminists. Now which is more probable?

- (a) Linda is a bank teller
- (b) Linda is a bank teller and a feminist

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Sequential updating

Proposition

Let $P(B \cap C) > 0$. Condition on B , then on C :

$$\tilde{P}(A) = P(A \mid B), \quad \tilde{P}(A \mid C) = P(A \mid B, C).$$

proof:

updating twice = updating on all the evidence at once

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The gambler's fallacy

back to roulette: is the wheel due for black?

$$P(\text{10th red} \mid \text{first 9 red}) =$$

- ▶ same as $P(\text{10th red})$
- ▶ wheel has no memory (**gambler's fallacy**)

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Summary

- ▶ new information $\Rightarrow$ **condition**: $P(A \mid B) = P(A \cap B)/P(B)$
- ▶ joint probability from easy conditionals $\Rightarrow$ **multiplication rule**
- ▶ $P(\cdot \mid B)$ is a probability function; all old rules apply
- ▶ be careful what you condition on (boy-girl, roulette)