

Stats 117 Lecture 6

Independence

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Organization

- ▶ definition of independence
 - ▶ Mendel's peas
- ▶ many independent events
- ▶ examples
 - ▶ Cardano, quality control
 - ▶ Sally Clark
- ▶ conditional independence

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Independence

Definition

Events A and B with positive probability are *independent* if

$$P(A \mid B) = P(A).$$

► equivalent: $P(B \mid A) = P(B)$. why?

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Multiplication rule for independent events

Proposition (definition?)

Events A and B are independent if and only if

$$P(A \cap B) = P(A)P(B).$$

proof:

► check: if $P(A) = 0$ or $P(A) = 1$, then what?

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Red and even

spin a roulette wheel once (38 pockets)

► 18 red, 18 even, 8 both red and even

$$P(\text{red} \mid \text{even}) =$$

$$P(\text{red}) =$$

independent?

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Disjoint events are super non-independent

if A and B are disjoint, with $P(A) > 0$ and $P(B) > 0$,

$$P(A \mid B) =$$

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Complements

Proposition

If A and B are independent, so are A and B^c .

proof:

- ▶ so same with A^c and B , and A^c and B^c

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Mendel's law of independent assortment

- ▶ two genes on different chromosomes; both parents $AaBb$
- ▶ $P(\text{albino}) = \frac{1}{4}$, $P(\text{freckled}) = \frac{1}{4}$
- ▶ genes on different chromosomes are inherited independently

	freckled	not freckled
albino		
not albino		

ratio 9 : 3 : 3 : 1; Mendel's peas: 315 : 108 : 101 : 32

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Many independent events

Definition

Events $A_1, A_2, \dots$ are *independent* if for every i and every finite collection $j_1, \dots, j_k \neq i$,

$$P(A_i \mid A_{j_1}, \dots, A_{j_k}) = P(A_i).$$

Theorem

$A_1, A_2, \dots$ are *independent* if and only if for every finite collection $i_1 < \dots < i_n$,

$$P(A_{i_1}, \dots, A_{i_n}) = P(A_{i_1}) \cdots P(A_{i_n}).$$

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Cardano, revisited

Cardano claimed $P(\text{at least one six in 3 rolls}) = 3 \cdot \frac{1}{6}$

question: how many rolls n for $P(\text{at least one six}) \geq \frac{1}{2}$?

$$P(\text{at least one six in } n \text{ rolls}) = 1 - P(\text{no sixes}) =$$

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Quality control

- ▶ batch of 50 medical devices
- ▶ each fails independently with probability p
- ▶ want $P(\text{no failures}) \geq .99$

$$P(\text{no failures}) =$$

how small must p be?

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Independence as a design principle

FM (frequency modulation) vs. AM (amplitude modulation) radio

- ▶ want radio signals that work during storms

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Cautionary tale: Sally Clark

- ▶ two of Sally Clark's infant sons died suddenly
- ▶ expert witness: $P(\text{SIDS death}) \approx \frac{1}{8543}$ for a family like hers,

$$P(\text{two SIDS deaths}) = \left(\frac{1}{8543} \right)^2 \approx \frac{1}{73 \text{ million}}$$

- ▶ convicted of murder in 1999

what went wrong?

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Conditional independence

Definition

Let $P(B) > 0$. Events $A_1, A_2, \dots$ are *conditionally independent given B* if they are independent under $P(\cdot \mid B)$:

$$P(A_{i_1}, \dots, A_{i_n} \mid B) = P(A_{i_1} \mid B) \cdots P(A_{i_n} \mid B).$$

- ▶ **neither** independence nor conditional independence implies the other

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COVID-19 testing

nurse gives a patient two tests

- ▶ $I = \{\text{infected}\}$, $T_i = \{\text{test } i \text{ positive}\}$,
 $C_i = \{\text{test } i \text{ correct}\}$

	independent?	given I ?
C_1, C_2		
T_1, T_2		

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COVID-19 testing: several nurses

three nurses: N_1, N_2 experts, N_3 inexperienced; one chosen at random gives both tests

- ▶ C_1, C_2 given N_1 :

- ▶ C_1, C_2 given N_1^c :

consequence: conditional independence given B says nothing about given B^c

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Summary

- ▶ independent: $P(A \mid B) = P(A)$, or $P(A \cap B) = P(A)P(B)$
- ▶ many events: multiply for every subcollection
- ▶ independence must be justified (Sally Clark)
- ▶ conditional independence: independence under $P(\cdot \mid B)$