

# Stats 117 Lecture 7

## Law of Total Probability and Bayes' Rule

Prof. John Duchi

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7-1

### Organization

- ▶ law of total probability
  - ▶ COVID tests, amoebas
  - ▶ Simpson's paradox
- ▶ Bayes' rule
  - ▶ positive COVID tests
  - ▶ spam filters

7-2

# Partitions

## Definition

Events  $B_1, B_2, \dots$  *partition*  $\Omega$  if they are disjoint and

$$\bigcup_{i \geq 1} B_i = \Omega.$$

picture:

7-3

## Law of total probability

### Theorem

Let  $B_1, B_2, \dots$  partition  $\Omega$ , with  $P(B_i) > 0$ . Then for any event  $A$ ,

$$P(A) = \sum_i P(B_i)P(A \mid B_i).$$

proof:

►  $P(A)$  is a weighted average of the  $P(A \mid B_i)$

7-4

## How accurate is a COVID test?

random New York City resident, March 2020

- ▶  $I = \{\text{infected}\}$ ,  $T = \{\text{test positive}\}$ ,  $C = \{\text{test correct}\}$
- ▶  $P(I) = .0005$
- ▶  $P(T | I) = .80$ ,  $P(T^c | I^c) = .99$

$$P(C) =$$

7-5

## Amoebas

each minute, an amoeba is equally likely to

- ▶ die ( $D$ ), live on ( $L$ ), or split in two ( $S$ )

start with one amoeba;  $E = \{\text{population dies out}\}$ ,  $p = P(E)$

$$p =$$

7-6

## Simpson's paradox: the death penalty

murder cases:

| defendant | cases | death penalty |
|-----------|-------|---------------|
| Black     | 191   | 7.9%          |
| White     | 483   | 11.0%         |

split by race of victim:

| defendant | Black victim |       | White victim |       |
|-----------|--------------|-------|--------------|-------|
|           | cases        | death | cases        | death |
| Black     | 143          | 2.8%  | 48           | 22.9% |
| White     | 16           | 0%    | 467          | 11.3% |

which defendants are more likely to get the death penalty?

7-7

## Simpson's paradox: explanation

law of total probability, partitioning by victim's race:

$$P(\text{death} \mid \text{Black}) =$$

$$P(\text{death} \mid \text{White}) =$$

- ▶ same rates, different weights
- ▶ most murders are within race; white victims more often lead to death penalty

7-8

## Bayes' rule

### Theorem

For events  $A$  and  $B$  with positive probability,

$$P(B | A) = \frac{P(B)P(A | B)}{P(A)}.$$

proof:

with the law of total probability,

$$P(B | A) = \frac{P(B)P(A | B)}{P(B)P(A | B) + P(B^c)P(A | B^c)}$$

7-9

## Testing positive

same New York resident tests positive. Infected?

►  $P(I) = .0005, \quad P(T | I) = .80, \quad P(T | I^c) = .01$

$$P(I | T) =$$

- $P(T | I) \neq P(I | T)$
- most positives come from the many uninfected

7-10

## Two positive tests

tests  $T_1, T_2$  conditionally independent given  $I$ , and given  $I^c$

all at once:

$$P(I \mid T_1, T_2) =$$

one at a time: new prior  $P(I \mid T_1) \approx .0385$

$$P(I \mid T_1, T_2) =$$

7-11

## Naive Bayes: spam filtering

email says “win free money!”

|                                   | win   | free | money |
|-----------------------------------|-------|------|-------|
| $P(\text{word} \mid \text{spam})$ | .02   | .05  | .1    |
| $P(\text{word} \mid \text{ham})$  | .0005 | .001 | .01   |

- ▶  $P(\text{spam}) = .1$
- ▶ *naïve*: words conditionally independent given label

$$\frac{P(\text{spam} \mid \text{email})}{P(\text{ham} \mid \text{email})} =$$

7-12

## Summary

- ▶ law of total probability:  $P(A) = \sum_i P(B_i)P(A | B_i)$
- ▶ averages can reverse subgroup trends (Simpson)
- ▶ Bayes' rule: reverse the conditioning,  
 $P(B | A) = P(B)P(A | B)/P(A)$
- ▶  $P(A | B) \neq P(B | A)$