

Stats 117 Lecture 8

Random Variables

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Organization

- ▶ random variables
- ▶ probability mass functions (p.m.f.s)
- ▶ cumulative distribution function (CDFs)
- ▶ examples
 - ▶ Bernoulli
 - ▶ binomial
 - ▶ geometric

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Random variables

Definition

A *random variable* is a function $X : \Omega \rightarrow \mathbf{R}$ (a *measurement*)

three coin tosses: $\Omega = \{HHH, HHT, \dots, TTT\}$

- ▶ $X(\omega)$ = number of heads
- ▶ $Y(\omega)$ = number of tails

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Events and random variables

important words (and type checks and mind bends):

- ▶ random variable X is *not* event
- ▶ events are of form

$$\{X \in C\} = \{\omega \in \Omega \mid X(\omega) \in C\}$$

or

$$\{X = x\} = \{\omega \in \Omega \mid X(\omega) = x\}$$

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Roulette payout

bet \$1 on 23; $\Omega = \{0, 00, 1, \dots, 36\}$

$$X(\omega) = \begin{cases} 35 & \omega = 23 \\ -1 & \text{otherwise} \end{cases}$$

► $P(X > 0) =$

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Discrete random variables

Definition

X is *discrete* if there is a finite or countable set $\{x_1, x_2, \dots\}$ with

$$\sum_i P(X = x_i) = 1.$$

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Probability mass function

Definition

The *probability mass function* of a discrete random variable X is

$$p_X(x) := P(X = x).$$

- ▶ $p_X(x) \geq 0$
- ▶ $\sum_x p_X(x) = 1$

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pmf: three coin tosses

X = number of heads

x	0	1	2	3
$p_X(x)$				

- ▶ Y = number of tails has *same* pmf

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Cumulative distribution function

Definition

The *cumulative distribution function* (cdf) of X is

$$F_X(t) := P(X \leq t) = \sum_{i: x_i \leq t} p_X(x_i).$$

- ▶ non-decreasing, continuous from right
- ▶ $F_X(t) \rightarrow 0$ as $t \rightarrow -\infty$, $F_X(t) \rightarrow 1$ as $t \rightarrow \infty$

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CDF: three coin tosses

picture:

- ▶ $P(X \geq 2) =$

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PMF from CDF

jumps of F_X give PMF:

$$p_X(x) = F_X(x) - \lim_{t \uparrow x} F_X(t)$$

- ▶ PMF and CDF carry same information

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Bernoulli

Definition

$X \sim \text{Bernoulli}(p)$: $X \in \{0, 1\}$ with

$$P(X = 1) = p_X(1) = p, \quad P(X = 0) = p_X(0) = 1 - p.$$

- ▶ single binary trial

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Binomial

Definition

$X \sim \text{Bin}(n, p)$ counts successes in n independent trials, each with success probability p :

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \dots, n.$$

why:

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Newton–Pepys

which is most likely?

- ▶ A : at least one six in 6 dice
- ▶ B : at least two sixes in 12 dice
- ▶ C : at least three sixes in 18 dice

$$P(A) \approx .6651, \quad P(B) \approx .6187, \quad P(C) \approx .5973$$

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Geometric

Definition

$N \sim \text{Geometric}(p)$ counts trials up to and including the first success:

$$f_N(n) = (1 - p)^{n-1}p, \quad n = 1, 2, \dots$$

► $P(N > n) =$

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Geometric: magic cards

- probability $1/6$ that one card in pack is “mythic rare”
- N = number of packs until get a mythic

$$P(N > 6) =$$

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Geometric: craps

point is five; roll until a five (win) or a seven (lose)

- ▶ $P(\text{five}) = \frac{4}{36}$, $P(\text{seven}) = \frac{6}{36}$, game ends with probability $\frac{10}{36}$
- ▶ $N = \text{number of rolls}$, $N \sim \text{Geometric}(10/36)$
- ▶ $P(N > 4) =$