

Stats 117 Lecture 9

Expected Value

Prof. John Duchi

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Organization

- ▶ definition
- ▶ interpretations
- ▶ examples: roulette, pooled testing, binomial
- ▶ paradoxes
- ▶ tail sum formula

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Expected value

Definition

The *expected value* of a discrete random variable X is

$$\mathbf{E}[X] := \sum_x x p_X(x) = \sum_x x P(X = x).$$

- weighted average of the values, weights the probabilities

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Six-sided die

faces 1, 2, 3, 4, 5, 6

- $\mathbf{E}[X] =$

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Interpretations

- ▶ long-run average:

$$\frac{X_1 + \cdots + X_n}{n} \rightarrow \mathbf{E}[X]$$

- ▶ center of mass: balance point of the pmf

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Roulette

bet \$1 on 23 (payout S), versus \$1 on red (payout R)

s	-1	35
$p_S(s)$	$\frac{37}{38}$	$\frac{1}{38}$

r	-1	1
$p_R(r)$	$\frac{20}{38}$	$\frac{18}{38}$

- ▶ $\mathbf{E}[S] = \mathbf{E}[R] = -\frac{1}{19} \approx -.053$

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Pooled testing

pool 10 samples; test the pool, and if positive test all 10 individually

- ▶ each sample positive with probability p , independently
- ▶ T = number of tests used
- ▶ $P(\text{pool positive}) =$

- ▶ $\mathbf{E}[T] =$

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When does pooling help?

want $\mathbf{E}[T] < 10$:

- ▶ $p < 1 - (.1)^{1/10} \approx .206$

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Binomial¹

if $X \sim \text{Bernoulli}(p)$ then $\mathbf{E}[X] = p$, and if $X \sim \text{Bin}(n, p)$ then

$$\mathbf{E}[X] = np$$

proof:

¹this is the hard way; later we will do it easy ways

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Pascal's wager

believe or not; God exists with probability $p > 0$

	God exists	God does not
believe	∞	$-c$
do not	$-\infty$	c

- expected payoff of believing is ∞ for any $p > 0$

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St. Petersburg paradox

pot starts at \$1 and doubles on each tails; you take the pot when heads appears

x	1	2	4	$\dots$
$p_X(x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\dots$

► $\mathbf{E}[X] =$

what would you pay to play? (but see homework...)

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Tail sum formula

Proposition

If X takes values in $\{0, 1, 2, \dots\}$, then

$$\mathbf{E}[X] = \sum_{x=0}^{\infty} P(X > x).$$

why:

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Geometric

$N \sim \text{Geometric}(p)$, so $P(N > n) = (1 - p)^n$

► $\mathbf{E}[N] =$

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Geometric: craps

point is five; game ends with probability $\frac{10}{36}$ each roll

► $N \sim \text{Geometric}(10/36)$

► $\mathbf{E}[N] =$

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