

Stats 117 Lecture 10

Transformations

Prof. John Duchi

Autumn 2026

10-1

Organization

- ▶ functions of a random variable
 - ▶ one-to-one
 - ▶ many-to-one
 - ▶ the fundamental correspondence
- ▶ examples: roulette, options
- ▶ random variables versus pmfs
- ▶ functions of multiple random variables: sums

10-2

Transformations

X a random variable, $g : \mathbf{R} \rightarrow \mathbf{R}$ a function

$$Y = g(X), \quad \text{i.e.} \quad Y(\omega) = g(X(\omega))$$

- ▶ Y is also a random variable
- ▶ question: what is the pmf of Y ?

10-3

Roulette

bet \$1 on 23; $I = \mathbf{1} \{\text{ball lands on 23}\}$

- ▶ $I \sim \text{Bernoulli}(1/38)$
- ▶ payout $S =$

i	0	1
$p_I(i)$	$\frac{37}{38}$	$\frac{1}{38}$

s	
$p_S(s)$	

10-4

Roulette: a bigger bet

bet \$10 on 23: $Y = 10S$

y	
$p_Y(y)$	

- probabilities do not change, only their locations

10-5

One-to-one transformations

Proposition

If g is one-to-one (each y comes from at most one x), then $Y = g(X)$ has pmf

$$p_Y(y) = p_X(g^{-1}(y)).$$

why:

10-6

Many-to-one transformations

Proposition

For any g , $Y = g(X)$ has pmf

$$p_Y(y) = P(g(X) = y) = \sum_{x: g(x)=y} p_X(x).$$

- add up probability of every x that lands on y

10-7

Call option

right to buy a stock for \$55 one week from now; X = stock price in one week

x	50	53	57	60
$p_X(x)$	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{4}{8}$	$\frac{1}{8}$

- value of option is $Y =$

10-8

Call option: pmf of Y

picture:



10-9

The fundamental bridge

Proposition

For any event A , the indicator I_A satisfies

$$P(A) = \mathbf{E}[I_A]$$

Example (Roulette)

Bet \$1 on even, outcome $X \in \{0, 00, 1, 2, \dots, 36\}$,

$$B = \mathbf{1}\{X \text{ even}\}$$

10-10

The map is not the territory

transform the random variable, *not* the pmf

► $Y = n - X$, but

$$p_Y(y) \neq n - p_X(y)$$

► $Y = 2X$, but

$$p_Y(y) \neq 2p_X(y)$$

type check:

10-11

Functions of multiple random variables

X, Y random variables on the same Ω ; $T = g(X, Y)$

► most important: $S = X + Y$

Proposition (Convolution identity)

If X and Y independent,

$$p_S(s) = \sum_x P(X = x, Y = s - x) = \sum_x p_X(x) p_Y(s - x)$$

10-12

Sum of independent binomials

Proposition

If $X \sim \text{Bin}(n, p)$ and $Y \sim \text{Bin}(m, p)$ are independent, then

$$X + Y \sim \text{Bin}(n + m, p).$$

story:

10-13

Sum of independent binomials: algebra

$$S = X + Y,$$

$$p_S(k) = \sum_i \binom{n}{i} p^i (1-p)^{n-i} \binom{m}{k-i} p^{k-i} (1-p)^{m-k+i}$$

► Vandermonde: $\sum_{i=0}^k \binom{n}{i} \binom{m}{k-i} = \binom{n+m}{k}$

10-14

Independence matters

$X \sim \text{Bin}(n, p)$ and $Y = X$, so $Y \sim \text{Bin}(n, p)$

► $X + Y =$

► is $X + Y \sim \text{Bin}(2n, p)$?

10-15

Sum of Bernoullis

$B_1, \dots, B_n \stackrel{\text{iid}}{\sim} \text{Bernoulli}(p)$

$$X = B_1 + B_2 + \dots + B_n \sim \text{Bin}(n, p)$$

why:

► (foreshadowing: an easy way to get $\mathbf{E}[X] = np$)

10-16