

Stats 117 Lecture 11

LotUS and Variance

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Autumn 2026

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Organization

- ▶ law of the unconscious statistician
- ▶ expected utility
- ▶ variance and standard deviation
 - ▶ shortcut formula
 - ▶ linear (affine) transformations

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Law of the unconscious statistician

Theorem

For any function g ,

$$\mathbf{E}[g(X)] = \sum_x g(x) p_X(x).$$

► no need for the pmf of $g(X)$

proof: group the values x with $g(x) = y$

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Call option

X = stock price in one week, strike \$55, so $Y = \max(X - 55, 0)$

x	50	53	57	60
$p_X(x)$	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{4}{8}$	$\frac{1}{8}$

► $\mathbf{E}[Y] =$

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Linear transformations

Proposition

$$\mathbf{E}[aX + b] = a\mathbf{E}[X] + b.$$

roulette: $B \sim \text{Bernoulli}(1/38)$ and $S = 36B - 1$

► $\mathbf{E}[S] =$

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Utility: concave functions

- human utility $u(w)$ for wealth w typically *concave*
- examples: $u(w) = \log w$, $u(w) = \sqrt{w}$, $u(w) = w^{1/4}$, ...

Theorem (Jensen's inequality)

For any concave function u and random variable W ,

$$\mathbf{E}[u(W)] \leq u(\mathbf{E}[W])$$

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Expected utility: roulette

bankroll \$10, utility $u(w) = \sqrt{w}$

- ▶ single number: $\mathbf{E}[\sqrt{10 + S}] \approx 3.098$
- ▶ red: $\mathbf{E}[\sqrt{10 + R}] \approx 3.15$
- ▶ same $\mathbf{E}[\text{money}]$, different $\mathbf{E}[\sqrt{\text{money}}]$

illustration:

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Variance

Definition

$$\text{Var}(X) := \mathbf{E}[(X - \mathbf{E}[X])^2], \quad \text{SD}(X) := \sqrt{\text{Var}(X)}.$$

- ▶ spread about the mean
- ▶ SD is in same units as X

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Roulette: two bets

both have $\mathbf{E}[\text{profit}] = -\frac{1}{19}$

s	-1	35
$p_S(s)$	$\frac{37}{38}$	$\frac{1}{38}$

r	-1	1
$p_R(r)$	$\frac{20}{38}$	$\frac{18}{38}$

► $\text{Var}(S) \approx 33.21, \quad \text{Var}(R) \approx .997$

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Shortcut formula

Proposition

$$\text{Var}(X) = \mathbf{E}[X^2] - (\mathbf{E}[X])^2.$$

proof: expand square

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Roulette, again

by the shortcut formula

► $\mathbf{E}[S^2] =$

► $\text{Var}(S) =$

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Binomial variance

Proposition

If $X \sim \text{Bin}(n, p)$, then $\text{Var}(X) = np(1 - p)$.

painful proof:

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Linear transformations of variance

Proposition

$$\text{Var}(aX + b) = a^2 \text{Var}(X).$$

- ▶ shifting does not change spread

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Third round (...) with roulette

$$S = 36B - 1, B \sim \text{Bernoulli}(1/38)$$

- ▶ $\text{Var}(B) =$

- ▶ $\text{Var}(S) =$

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