

Stats 117 Lecture 14

Expectations of Multiple Random Variables

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14-1

Organization

- ▶ 2D LotUS
- ▶ expectation of a sum: linearity
 - ▶ indicators: binomial and hypergeometric the easy way
 - ▶ hire and fire
- ▶ expectation of a product

14-2

2D LotUS

Theorem

For any function g of two variables,

$$\mathbf{E}[g(X, Y)] = \sum_x \sum_y g(x, y) p_{X,Y}(x, y).$$

- ▶ no need for pmf of $g(X, Y)$
- ▶ same proof as before: group (x, y) with $g(x, y) = z$

14-3

Linearity of expectation

Theorem

For any random variables X and Y ,

$$\mathbf{E}[X + Y] = \mathbf{E}[X] + \mathbf{E}[Y].$$

- ▶ **no** independence required
- ▶ with $\mathbf{E}[aX + b] = a\mathbf{E}[X] + b$: $\mathbf{E}[aX + bY] = a\mathbf{E}[X] + b\mathbf{E}[Y]$

proof: 2D LotUS with $g(x, y) = x + y$

14-4

Linearity: proof

$$\mathbf{E}[X + Y] = \sum_x \sum_y (x + y) p_{X,Y}(x, y)$$

14-5

The indicator method

to find $\mathbf{E}[X]$ for a count X :

1. write $X = I_1 + I_2 + \cdots + I_n$, each I_k an indicator
2. $\mathbf{E}[I_k] =$
3. add them up

why it is easy: no joint pmf, no independence, no sums over $\binom{n}{k}$

14-6

Binomial (easy way)

$X \sim \text{Bin}(n, p)$ counts successes in n trials

► $I_k = \mathbf{1} \{\text{trial } k \text{ a success}\} \sim \text{Bernoulli}(p)$

► $X =$

► $\mathbf{E}[X] =$

compare sum $\sum_k k \binom{n}{k} p^k (1-p)^{n-k}$ from Lecture 9

14-7

Hypergeometric (easy way)

N cans, M defective, draw n without replacement

$$Y \sim \text{HyperGeom}(n, M, N)$$

► $I_k = \mathbf{1} \{k\text{th can drawn is defective}\}$

► $\mathbf{E}[I_k] =$

► $\mathbf{E}[Y] =$

► $I_1, \dots, I_n$ **not** independent

14-8

Hire and fire

manager interviews n candidates in random order, hires any candidate better than everyone seen so far (firing incumbent)

- ▶ X = number of people hired
- ▶ $I_k = \mathbf{1}\{\text{candidate } k \text{ is hired}\}$, $X = I_1 + \cdots + I_n$
- ▶ candidate k hired $\Leftrightarrow$

14-9

Hire and fire expectations

- ▶ $\mathbf{E}[I_k] = P(\text{candidate } k \text{ best of first } k) =$
- ▶ $\mathbf{E}[X] =$
- ▶ $\sum_{k=1}^n \frac{1}{k} \approx \int_1^n \frac{1}{x} dx = \log n$
- ▶ $n = 100$: about 5.2 hires

14-10

Expectation of a product

Theorem

If X and Y are independent, then

$$\mathbf{E}[XY] = \mathbf{E}[X] \mathbf{E}[Y],$$

and more generally $\mathbf{E}[g(X)h(Y)] = \mathbf{E}[g(X)] \mathbf{E}[h(Y)]$.

► independence **required**

proof: 2D LotUS, then factor $p_{X,Y} = p_X p_Y$

14-11

A converse

Theorem

If

$$\mathbf{E}[g(X)h(Y)] = \mathbf{E}[g(X)] \mathbf{E}[h(Y)]$$

for all g, h , then X and Y are independent.

14-12