

Stats 117 Lecture 19

General Expectations

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Organization

- ▶ discretizing random variables
- ▶ discretized expectations
- ▶ general definition of expectation
- ▶ continuous random variables
- ▶ examples, tricks, and median

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Thought experiment: discretize a random variable

for $s > 0$ and $x \in \mathbf{R}$, define s -discretization of x

$$[x]_s := s \cdot \lfloor x/s \rfloor$$

► shrinks x to largest multiple $k \cdot s \leq x$ (s for small / scale)

observation: for any random variable X ,

$$[X]_s$$

is discrete random variable with values

and

$$P([X]_s = k \cdot s) =$$

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Discretized expectation

expectation of discretized random variable:

$$\begin{aligned} \mathbf{E}[[X]_s] &= \sum_k k \cdot s \cdot P([X]_s = k \cdot s) \\ &= \sum_k k \cdot s \cdot P(k \cdot s \leq X < (k+1) \cdot s) \end{aligned}$$

question: what is the right expectation?

► remember:

$$\omega \longrightarrow X(\omega) \longrightarrow [X(\omega)]_s$$

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Bit precision

- ▶ bit precision of number
- ▶ let $s = 2^{-m}$ for $m = 0, 1, 2, \dots$
- ▶ define

$$x_m := [x]_{2^{-m}} = \lfloor 2^m x \rfloor / 2^m$$

- ▶ have $x_m \uparrow x$ as

$$x_m \leq x_{m+1} \leq x$$

bit-precision expectation:

$$\mathbf{E}[X_m] = \sum_k \frac{k}{2^m} P\left(\frac{k}{2^m} \leq X < \frac{k+1}{2^m}\right)$$

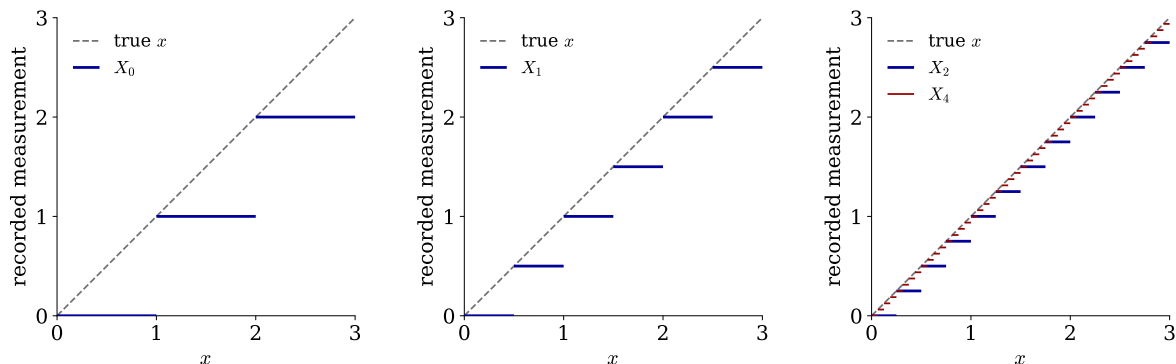
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Increasing resolution

- ▶ binary discretization

$$X_m = [X]_{2^{-m}} = \lfloor 2^m \cdot X \rfloor / 2^m$$

- ▶ remember: $\omega \mapsto X(\omega) \mapsto [X(\omega)]_{2^{-m}}$



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Discrete random variables

for discrete X , discretized expectation satisfies

$$\mathbf{E} \left[[X]_s \right] \leq \mathbf{E}[X] \leq \mathbf{E} \left[[X]_s \right] + s$$

why?

conclusion: take discretization scale $s \downarrow 0$,

$$\mathbf{E} [X] = \lim_{s \downarrow 0} \mathbf{E} \left[[X]_s \right]$$

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Example: uniform distribution

- ▶ density $f(u) = 1$ for $0 \leq u \leq 1$
- ▶ discretization level $s = 1/m$:

$$[u]_{1/m} \in \left\{ 0, \frac{1}{m}, \frac{2}{m}, \dots, \frac{m-1}{m}, 1 \right\}$$

- ▶ expectation

$$\mathbf{E} \left[[U]_{1/m} \right] =$$

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Bit precision (again)

- keep number bits after decimal (binary expansion)

$$X_m = [X]_{2^{-m}} = [X(\omega)]_{2^{-m}}$$

- m th expectation increases with m :

$$\mathbf{E}[X_{m-1}] \leq \mathbf{E}[X_m] \leq \mathbf{E}[X_{m+1}]$$

Definition

For any* random variable X ,

$$\mathbf{E}[X] := \lim_{m \rightarrow \infty} \mathbf{E}[X_m] = \lim_{m \rightarrow \infty} \mathbf{E}[[X]_{2^{-m}}]$$

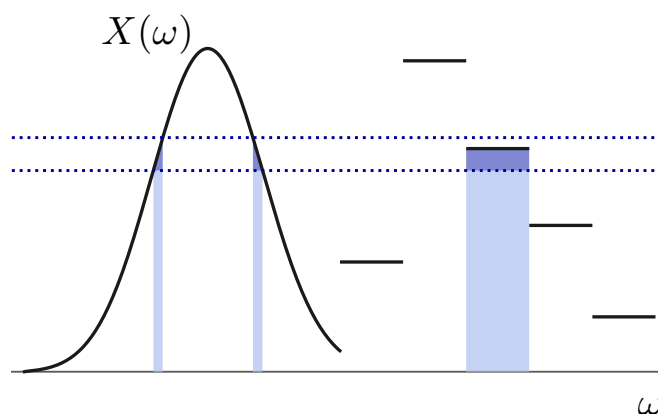
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General expectations

Definition

Let X be any* random variable. We define its expectation as the limiting discretized expectation

$$\mathbf{E}[X] := \lim_{s \downarrow 0} \mathbf{E}[[X]_s].$$



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Expectations of continuous random variables

Theorem

Let X have density $f(x)$. Then

$$\mathbf{E}[X] = \int x f(x) dx.$$

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Example: radioactive particles

- ▶ radioactive particle has decay time $T \sim \text{Exp}(1.2)$
- ▶ density

$$f(t) = \begin{cases} 1.2e^{-1.2t} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

computing expectation $\mathbf{E}[T] = 1/1.2$

$$\mathbf{E}[T] = \int_0^{\infty} t \cdot f(t) dt$$

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A streetfighting trick: tail “sums”

can *always* use the CDF to compute expectations

Proposition

If $X \geq 0$ is a nonnegative random variable, then

$$\mathbf{E}[X] = \int_0^\infty P(X \geq t) dt$$

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Example: radioactive particles (again)

- ▶ radioactive particle has decay time $T \sim \text{Exp}(1.2)$
- ▶ density

$$f(t) = \begin{cases} 1.2e^{-1.2t} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

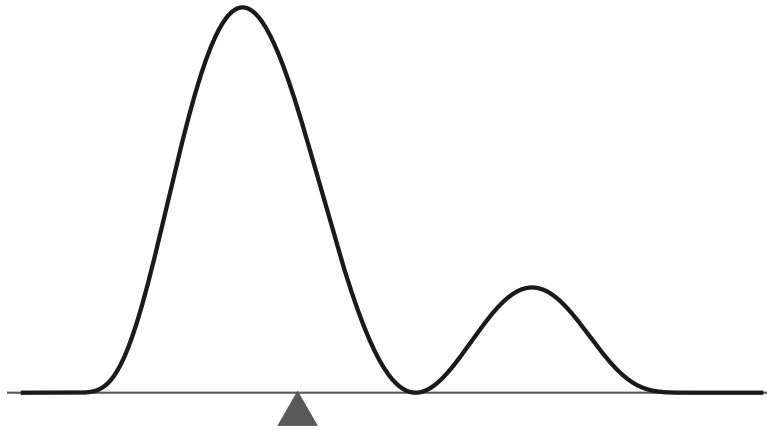
computing expectation $\mathbf{E}[T] = 1/1.2$

$$\mathbf{E}[T] = \int_0^\infty P(T \geq t) dt$$

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Interpretation: angular moments

- $\mathbf{E}[X]$ is balance point of $f(x)$



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Another summary statistic

Definition

A *median* of a continuous r.v. X is value m such that

$$\frac{1}{2} = P(X \geq m) \quad \text{and} \quad \frac{1}{2} = P(X \leq m)$$

not the same as the mean: compare $T \sim \text{Exp}(1.2)$

median	mean
$\frac{\log 2}{1.2}$	$\frac{1}{1.2}$

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Sometimes the mean doesn't exist

- ▶ Cauchy distribution on $x \in \mathbf{R}$ has density

$$f(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}$$

- ▶ what is $\mathbf{E}[X]$?