# STATS 218 Homework 1

Due date: Monday, Apr 4

## Problem 1

Let $(a_n)_{n\geq 1}$ be a sequence of numbers such that $a_n \geq 0$ for all $n$, and $\lim_{n\to\infty} a_n = 0$. Consider random variables $X_n \sim \mathsf{N}(0, a_n)$, not necessarily independent.

(i) Is there a random variable $X_\infty$ such that $X_n \xrightarrow{p} X_\infty$? Justify your answer.

(ii) Is there a random variable $X_\infty$ such that $X_n \xrightarrow{d} X_\infty$? Justify your answer.

(iii) Let $q \geq 1$. Is there a random variable $X_\infty$ such that $X_n \xrightarrow{L^q} X_\infty$? Justify your answer.

(iv) Consider the case $a_n = 1/n$. Is there $X_\infty$ such that $X_n \xrightarrow{a.s.} X_\infty$? Justify your answer.

(v) Repeat point (iv) for $X_n$ independent and $a_n = 1/\log\log n$.

## Problem 2

Find random variables $X$ and $(X_n)_{n\geq 1}$ such that $\lim_{n\to\infty} \mathbb{E}[|X_n - X|^2] = 0$, but $\mathbb{E}[X_n^2] = \infty$ for all $n$.

## Problem 3

In defining a renewal process we suppose that $F(\infty)$, the probability that an interarrival time is finite, equals 1. If $F(\infty) < 1$, then after each renewal there is a positive probability $1 - F(\infty)$ that there will be no further renewals. Argue that when $F(\infty) < 1$ the total number of renewals, call it $N(\infty)$, is such that $1 + N(\infty)$ has a geometric distribution with mean $1/(1 - F(\infty))$.

## Problem 4

Express in words what the random variable $X_{N(t)+1}$ represents (*Hint*: It is the length of which renewal interval?) Show that

$$P\left\{X_{N(t)+1} \geq x\right\} \geq \bar{F}(x).$$

Compute the above exactly when $F(x) = 1 - e^{-\lambda x}$.

## Problem 5

Prove the renewal equation

$$m(t) = F(t) + \int_0^t m(t-x)dF(x).$$