STATS 218 Homework 3

Due date: Monday, Apr 18

Problem 1 (Ross Ex. 3.14)

Let $C(t)$ and $E(t)$ denote the age and excess at t of a renewal-process. Fill in the missing terms:

(a) $C(t) > x \leftrightarrow 0$ events in the interval ______ ?

(b) $E(t) > x \leftrightarrow 0$ events in the interval ______ ?

(c) $P\{E(t) > x\} = P\{C(\quad) > \quad\}$.

(d) Compute the joint distribution of $C(t)$ and $E(t)$ for a Poisson process.

Problem 2 (Ross Ex. 3.17)

An equation of the form

$$g(t) = h(t) + \int_0^t g(t-x)dF(x)$$

is called a renewal-type equation. In convolution notation the above states that

$$g = h + g * F.$$

Either iterate the above or use Laplace transforms to show that a renewal-type equation has the solution

$$g(t) = h(t) + \int_0^t h(t-x)dm(x),$$

where $m(x) = \sum_{n=1}^{\infty} F_n(x)$. If h is directly Riemann integrable and F nonlattice with finite mean, one can then apply the key renewal theorem to obtain

$$\lim_{t\to\infty} g(t) = \frac{\int_0^\infty h(t)dt}{\int_0^\infty \bar{F}(t)dt}.$$

Renewal-type equations for $g(t)$ are obtained by conditioning on the time at which the process probabilistically starts over. Obtain a renewal-type equation for:

(a) $P(t)$, the probability an alternating renewal process is on at time t;

(b) $g(t) = E[C(t)]$, the expected age of a renewal process at t.

Apply the key renewal theorem to obtain the limiting values in (a) and (b).
Note Alternating renewal process is defined in Ross Section 3.4.1, $F_n(t) = P(S_n \leq t)$.

Problem 3

Consider a Poisson point process, that is a renewal process with $P(X_1 > t) = \exp(-\lambda t)$.

(1) Apply the formula for $q_*(z) = \lim_{t\to\infty} P(E(t) > z)$, and compute this limit.

(2) Write the integral equation for $P(E(t) > z)$ derived in class (see also Section 10.3 of Grimmett). Show that $P(E(t) > z) = q_*(z)$ for all t is a solution of this equation. What can you deduce?