# STATS 218 Homework 5

Due date: Monday, May 9

Let $Z_n$ be the population size after $n$ generations for a branching process with offspring distribution $P_X$. Namely, for each $n \geq 1$, let $(X_{n,k})_{k\geq 1}$ be i.i.d. random variables $X_{n,k} \sim P_X$ independent of $(X_{n',k})_{n'<n}$. For $n = 0$, set $Z_0 = 1$. For $n \geq 1$, set

$$Z_n = \sum_{k=1}^{Z_{n-1}} X_{n,k}.$$

Assume $\mu = \mathbb{E}[X_{n,k}] < \infty$, $v = \mathrm{Var}(X_{n,k}) < \infty$.

(1) Define $M_n := \mu^{-n} Z_n$. Prove that $M_n \overset{a.s.}{\to} M_\infty$ for a random variable $M_\infty$.

(2) Consider the case $\mu < 1$. What does point (1) imply about $\lim_{n\to\infty} Z_n$? What is the distribution of $M_\infty$ in this case?

(3) Consider next $\mu = 1$, and answer the same questions as in point (2).

(4) For $\mu > 1$, show that $\sup_n \mathbb{E}(M_n^2) \leq C < \infty$ for some constant $C$. What does this imply about $\mathbb{E}(M_\infty)$?

(5) Show that (always for $\mu > 1$) the previous points imply

$$\mathbb{P}\left(\lim_{n\to\infty} Z_n = \infty\right) > 0.$$

(the population diverges with positive probability)

(6) For $X \sim P_X$, define ($s \in \mathbb{R}_{>0}$)

$$G(s) := \mathbb{E}[s^X] = \sum_{k=0}^{\infty} \mathbb{P}(X = k) s^k.$$

Assume $\mu > 1$ and $\mathbb{P}(X = 0) > 0$. Show that there exists a unique $s_* \in (0, 1)$ such that $s_* = G(s_*)$.

(7) Define $Y_n = s_*^{Z_n}$. Show that $Y_n$ is a martingale and prove that $Y_n \overset{a.s.}{\to} Y_\infty$ for some random variable $Y_\infty \in [0, 1]$.

(8) Argue that $Y_\infty \in \{0, 1\}$, i.e., $Y_\infty$ cannot take values in $(0, 1)$ with positive probability.

(9) Deduce that

$$\mathbb{P}\left(\lim_{n\to\infty} Z_n = 0\right) = s_*.$$