STATS 218 Homework 1 Solutions

Kangjie Zhou

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Problem 1

Let $(a_n)_{n\geq 1}$ be a sequence of numbers such that $a_n \geq 0$ for all n, and $\lim_{n\to\infty} a_n = 0$. Consider random variables $X_n \sim \mathsf{N}(0, a_n)$, not necessarily independent.

(i) Is there a random variable X_∞ such that $X_n \xrightarrow{p} X_\infty$? Justify your answer.

(ii) Is there a random variable X_∞ such that $X_n \xrightarrow{d} X_\infty$? Justify your answer.

(iii) Let $q \geq 1$. Is there a random variable X_∞ such that $X_n \xrightarrow{L^q} X_\infty$? Justify your answer.

(iv) Consider the case $a_n = 1/n$. Is there X_∞ such that $X_n \xrightarrow{a.s.} X_\infty$? Justify your answer.

(v) Repeat point (iv) for X_n independent and $a_n = 1/\log\log n$.

Solution.

(i) Take $X_\infty = 0$. For any $\varepsilon > 0$, we have (Chebyshev's inequality)

$$\mathbb{P}(|X_n| \geq \varepsilon) \leq \frac{\mathbb{E}[X_n^2]}{\varepsilon^2} = \frac{a_n}{\varepsilon^2} \to 0 \Rightarrow X_n \xrightarrow{p} 0.$$

(ii) $X_n \xrightarrow{p} 0$ implies $X_n \xrightarrow{d} 0$.

(iii) Since $X_n \sim \mathsf{N}(0, a_n)$, we have $X_n \stackrel{d}{=} a_n^{1/2} Z$ where $Z \sim \mathsf{N}(0,1)$, thus leading to

$$\mathbb{E}[|X_n|^q] = a_n^{q/2}\mathbb{E}[|Z|^q] \to 0 \text{ as } n \to \infty.$$

Hence, $X_n \xrightarrow{L^q} 0$.

(iv) We will prove that $X_n \xrightarrow{a.s.} 0$. Let $\varepsilon_n = n^{-\alpha}$ for some $\alpha \in (0, 1/4)$, then we have

$$\sum_{n=1}^{\infty} \mathbb{P}\left(|X_n| \geq \varepsilon_n\right) \leq \sum_{n=1}^{\infty} \frac{\mathbb{E}[|X_n|^4]}{\varepsilon_n^4} = \sum_{n=1}^{\infty} \frac{3}{n^2 \varepsilon_n^4} = 3\sum_{n=1}^{\infty} \frac{1}{n^{2-4\alpha}} < \infty.$$

According to Borel-Cantelli Lemma, we have $\mathbb{P}(|X_n| \geq \varepsilon_n, \text{ i.o.}) = 0$. This further implies that with probability one, $|X_n| \leq \varepsilon_n$ for all large enough n. Hence, $X_n \xrightarrow{a.s.} 0$ almost surely.

(v) We prove that with probability one, X_n does not converge to 0. To this end, we fix an $\varepsilon > 0$ and show that

$$\sum_{n=1}^{\infty} \mathbb{P}\left(|X_n| \geq \varepsilon\right) = \infty.$$

Since the X_n's are mutually independent, applying Borel-Cantelli Lemma yields that $\mathbb{P}(|X_n| \geq \varepsilon, \text{ i.o.}) = 1$, which further implies the desired result.

Now we prove our claim. According to Mills' ratio, we have

$$\mathbb{P}\left(|X_n| \geq \varepsilon\right) = 2\mathbb{P}\left(X_n \geq \varepsilon\right) = 2\mathbb{P}\left(Z \geq \frac{\varepsilon}{\sqrt{a_n}}\right) \sim \sqrt{\frac{2}{\pi}} \frac{\sqrt{a_n}}{\varepsilon} \exp\left(-\frac{\varepsilon^2}{2a_n}\right)$$
$$= \sqrt{\frac{2}{\pi}} \frac{1}{\varepsilon\sqrt{\log\log n}} \left(\log n\right)^{-\varepsilon^2/2} \gg \frac{1}{n} \text{ for large } n.$$

As a consequence, we have $\sum_{n=1}^{\infty} \mathbb{P}(|X_n| \geq \varepsilon) = \infty$.

Problem 2

Find random variables X and $(X_n)_{n\geq 1}$ such that $\lim_{n\to\infty} \mathbb{E}[|X_n - X|^2] = 0$, but $\mathbb{E}[X_n^2] = \infty$ for all n.

Solution. Choose $X_n = X$ where $\mathbb{E}[X^2] = \infty$, e.g., X follows Cauchy distribution.

Problem 3

In defining a renewal process we suppose that $F(\infty)$, the probability that an interarrival time is finite, equals 1. If $F(\infty) < 1$, then after each renewal there is a positive probability $1 - F(\infty)$ that there will be no further renewals. Argue that when $F(\infty) < 1$ the total number of renewals, call it $N(\infty)$, is such that $1 + N(\infty)$ has a geometric distribution with mean $1/(1 - F(\infty))$.

Solution. By definition, we see that for $k \geq 1$,

$$\mathbb{P}\left(1 + N(\infty) = k\right) = \mathbb{P}\left(N(\infty) = k - 1\right) = F(\infty)^{k-1}\left(1 - F(\infty)\right).$$

Therefore, $1 + N(\infty)$ follows geometric distribution with success probability $1 - F(\infty)$.

Problem 4

Express in words what the random variable $X_{N(t)+1}$ represents (*Hint*: It is the length of which renewal interval?) Show that

$$P\left\{X_{N(t)+1} \geq x\right\} \geq \bar{F}(x).$$

Compute the above exactly when $F(x) = 1 - e^{-\lambda x}$.

Solution. $X_{N(t)+1}$ is the length of the renewal interval that contains t. Let $F_{S_{N(t)}}$ denote the CDF of $S_{N(t)}$, then we have

$$\mathbb{P}\left(X_{N(t)+1} \geq x\right) = \int_0^t \mathbb{P}\left(X_{N(t)+1} \geq x | S_{N(t)} = s\right) dF_{S_{N(t)}}(s)$$
$$= \int_0^t \mathbb{P}\left(X \geq x | X \geq t-s\right) dF_{S_{N(t)}}(s) = \int_0^t \frac{\bar{F}(\max(x, t-s))}{\bar{F}(t-s)} dF_{S_{N(t)}}(s).$$

For any $s \in [0, t]$, it follows that $\bar{F}(\max(x, t-s))/\bar{F}(t-s) \geq \bar{F}(x)$. Hence, $\mathbb{P}\left(X_{N(t)+1} \geq x\right) \geq \bar{F}(x)$.

When $F(x) = 1 - e^{-\lambda x}$, $\bar{F}(x) = e^{-\lambda x}$. According to Ross Lemma 3.4.3,

$$dF_{S_{N(t)}}(s) = \bar{F}(t)\delta(0) + \bar{F}(t-s)dm(s),$$

where $m(s) = \mathbb{E}[N(s)] = \lambda s$. This finally leads to

$$\mathbb{P}\left(X_{N(t)+1} \geq x\right) = \bar{F}(\max(x, t)) + \int_0^t \bar{F}(\max(x, t-s))\lambda ds$$
$$= \exp(-\lambda \max(x, t)) + \lambda \int_0^t \exp(-\lambda \max(x, t-s))ds$$
$$= \exp(-\lambda \max(x, t)) + \lambda \int_0^t \exp(-\lambda \max(x, s))ds,$$

which can be easily computed.

Problem 5

Prove the renewal equation

$$m(t) = F(t) + \int_0^t m(t-x)dF(x).$$

Solution. Using the law of total expectation, we obtain that

$$m(t) = \mathbb{E}[N(t)] = \mathbb{E}[\mathbb{E}[N(t)|X_1]] = \int_0^t \mathbb{E}[\mathbb{E}[N(t)|X_1 = x]]dF(x)$$
$$= \int_0^t (1 + \mathbb{E}[N(t-x)])dF(x) = F(t) + \int_0^t m(t-x)dF(x).$$