STATS 218 Homework 2 Solutions

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Problem 1 (Grimmett Ex. 10.2.2)

Let $Z_1, Z_2, \ldots$ be independent identically distributed random variables with mean 0 and finite variance σ^2, and let $T_n = \sum_{i=1}^{n} Z_i$. Let M be a finite stopping time with respect to the Z_i such that $\mathbb{E}(M) < \infty$. Show that $\operatorname{var}(T_M) = \mathbb{E}(M)\sigma^2$.
Solution. This is Wald's second identity. See the proof here (Theorem 12.6).

Problem 2 (Ross Ex. 3.9)

Consider a single-server bank in which potential customers arrive at a Poisson rate λ. However, an arrival only enters the bank if the server is free when he or she arrives. Let G denote the service distribution.

(a) At what rate do customers enter the bank?

(b) What fraction of potential customers enter the bank?

(c) What fraction of time is the server busy?

Solution.

(a) By definition, the interarrival times are i.i.d. with distribution $G * \text{Pois}(\lambda)$, which has expectation $\mu(G)+1/\lambda$ ($\mu(G)$ denotes the mean of G). The arrival rate should be $1/(\mu(G)+1/\lambda) = \lambda/(1+\lambda\mu(G))$.

(b) The customer enters the bank if and only if the server is not busy. According to Ross Theorem 3.4.4, the fraction is $(1/\lambda)/(\mu(G)+1/\lambda) = 1/(1+\lambda\mu(G))$.

(c) $1 - 1/(1+\lambda\mu(G)) = \lambda\mu(G)/(1+\lambda\mu(G))$.

Problem 3 (Ross Ex. 3.11)

Consider a miner trapped in a room that contains three doors. Door 1 leads her to freedom after two-days' travel; door 2 returns her to her room after four-days' journey, and door 3 returns her to her room after eight-days' journey. Suppose at all times she is equally to choose any of the three doors, and let T denote the time it takes the miner to become free.

(a) Define a sequence of independent and identically distributed random variables $X_1, X_2, \ldots$ and a stopping time N such that

$$T = \sum_{l=1}^{N} X_l.$$

Note You may have to imagine that the miner continues to randomly choose doors even after she reaches safety.

(b) Use Wald's equation to find $\mathbb{E}[T]$.

(c) Compute $\mathbb{E}\left[\sum_{t=1}^{N} X_t \mid N = n\right]$ and note that it is not equal to $\mathbb{E}\left[\sum_{t=1}^{n} X_t\right]$.

(d) Use part (c) for a second derivation of $\mathbb{E}[T]$.

Solution.

(a) Let the random variable X be such that

$$\mathbb{P}(X = 2) = \mathbb{P}(X = 4) = \mathbb{P}(X = 8) = \frac{1}{3},$$

and $X_1, X_2, \cdots \sim_{\text{i.i.d.}} X$, $N = \min\{n \in \mathbb{N} : X_n = 2\}$.

(b) It is easy to see that N follows geometric distribution with success probability $1/3$, hence $\mathbb{E}[N] = 3$. Wald's identity implies that $\mathbb{E}[T] = \mathbb{E}[N]\mathbb{E}[X] = 14$.

(c) By definition of N, we have

$$\mathbb{E}\left[\sum_{i=1}^{N} X_i \middle| N = n\right] = \mathbb{E}\left[\sum_{i=1}^{n} X_i \middle| X_1 \neq 2, \cdots, X_{n-1} \neq 2, X_n = 2\right]$$
$$= \sum_{i=1}^{n-1} \mathbb{E}\left[X_i | X_i \neq 2\right] + 2 = 6(n-1) + 2 = 6n - 4,$$

but $\mathbb{E}[\sum_{i=1}^{n} X_i] = 14n/3$.

(d) From part (c) we know that $\mathbb{E}[T|N] = 6N - 4$. Hence, $\mathbb{E}[T] = 6\mathbb{E}[N] - 4 = 14$.