STATS 218 Homework 3 Solutions

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Problem 1 (Ross Ex. 3.14)

Let $C(t)$ and $E(t)$ denote the age and excess at t of a renewal-process. Fill in the missing terms:

(a) $C(t) > x \leftrightarrow 0$ events in the interval ______ ?

(b) $E(t) > x \leftrightarrow 0$ events in the interval ______ ?

(c) $P\{E(t) > x\} = P\{C(\quad) > \quad\}$.

(d) Compute the joint distribution of $C(t)$ and $E(t)$ for a Poisson process.

Solution.

(a) $(t - x, t)$.

(b) $(t, t + x)$.

(c) $P(C(t + x) > x)$.

(d) For $x, y > 0$, we have

$$P(C(t) > x, E(t) > y) = P(0 \text{ events in } (t - x, t + y)) = \exp(-\lambda(x + y)),$$

where the last step follows from the memoryless property of exponential distribution.

Problem 2 (Ross Ex. 3.17)

An equation of the form

$$g(t) = h(t) + \int_0^t g(t - x) dF(x)$$

is called a renewal-type equation. In convolution notation the above states that

$$g = h + g * F.$$

Either iterate the above or use Laplace transforms to show that a renewal-type equation has the solution

$$g(t) = h(t) + \int_0^t h(t - x) dm(x),$$

where $m(x) = \sum_{n=1}^{\infty} F_n(x)$. If h is directly Riemann integrable and F nonlattice with finite mean, one can then apply the key renewal theorem to obtain

$$\lim_{t \to \infty} g(t) = \frac{\int_0^\infty h(t) dt}{\int_0^\infty \bar{F}(t) dt}.$$

Renewal-type equations for $g(t)$ are obtained by conditioning on the time at which the process probabilistically starts over. Obtain a renewal-type equation for:

(a) $P(t)$, the probability an alternating renewal process is on at time t;

(b) $g(t) = E[C(t)]$, the expected age of a renewal process at t.

Apply the key renewal theorem to obtain the limiting values in (a) and (b).
Note Alternating renewal process is defined in Ross Section 3.4.1, $F_n(t) = P(S_n \leq t)$.
Solution. By iterating the equation, we obtain that

$$\begin{aligned} g =& h + g * F = h + (h + g * F) * F = h + h * F + g * (F * F) \\ =& h + \sum_{i=1}^{n} h * (F^{*i}) + g * (F^{*(n+1)}) = h + \sum_{n=1}^{\infty} h * (F^{*n}), \end{aligned}$$

where F^{*n} denotes the n-fold self convolution of F. Note that $F^{*n} = F_n$, thus leading to

$$g = h + h * \left(\sum_{n=1}^{\infty} F_n \right) = h + h * m \Rightarrow g(t) = h(t) + \int_0^t h(t-x) dm(x).$$

(a) This is exactly Ross Theorem 3.4.4.

(b) Note that $C(t) = t - S_{N(t)}$. Conditioning on $S_{N(t)}$, we obtain that

$$g(t) = tP(S_{N(t)} = 0) + \int_0^x (t-x) dF_{S_{N(t)}}(x) = t\bar{F}(t) + \int_0^x (t-x)\bar{F}(t-x) dm(x).$$

This is a renewal equation with $h(t) = t\bar{F}(t)$. Applying the key renewal theorem yields that

$$\lim_{t\to\infty} g(t) = \frac{\int_0^\infty t\bar{F}(t)dt}{\int_0^\infty \bar{F}(t)dt} = \frac{E[X^2]}{2E[X]}, \text{ where } X \sim F.$$

Problem 3

Consider a Poisson point process, that is a renewal process with $P(X_1 > t) = \exp(-\lambda t)$.

(1) Apply the formula for $q_*(z) = \lim_{t\to\infty} P(E(t) > z)$, and compute this limit.

(2) Write the integral equation for $P(E(t) > z)$ derived in class (see also Section 10.3 of Grimmett). Show that $P(E(t) > z) = q_*(z)$ for all t is a solution of this equation. What can you deduce?

Solution.

(1) Since $F(t) = 1 - \exp(-\lambda t)$, we have $\mu = 1/\lambda$. Hence,

$$q_*(z) = \frac{1}{\mu} \int_z^\infty (x-z)\lambda \exp(-\lambda x) dx = \frac{\exp(-\lambda z)}{\mu\lambda} = \exp(-\lambda z).$$

(2) Fix z and denote $\mu(t) = P(E(t) > z)$, we have

$$\mu(t) = 1 - F(t+z) + \int_0^t \mu(t-x) dF(x).$$

Substituting $\mu(t) = q_*(z)$ into the above equation, we get that

$$\text{RHS} = \exp(-\lambda(t+z)) + \int_0^t \exp(-\lambda z)\lambda \exp(-\lambda x) dx = \exp(-\lambda z) = \text{LHS}.$$

This implies $P(E(t) > z) = q_*(z)$ is a constant not depending on t, which can be deduced from the memoryless property as well.