STATS 218 Homework 4 Solutions

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Problem 1 (Ross Ex. 6.1)

If $\{Z_n, n \geq 1\}$ is a martingale, show that, for $1 \leq k < n$,

$$E\left[Z_n \mid Z_1, \ldots, Z_k\right] = Z_k.$$

Solution. By Tower property of conditional expectation, we get

$$\begin{aligned} E[Z_n|Z_1, \cdots, Z_k] =& E[E[Z_n|Z_1, \cdots, Z_{n-1}]|Z_1, \cdots, Z_k] = E[Z_{n-1}|Z_1, \cdots, Z_k] \\ =& \cdots = E[Z_k|Z_1, \cdots, Z_k] = Z_k. \end{aligned}$$

Problem 2 (Ross Ex. 6.2)

For a martingale $\{Z_n, n \geq 1\}$, let $X_i = Z_i - Z_{i-1}, i \geq 1$, where $Z_0 \equiv 0$. Show that

$$\operatorname{Var}\left(Z_n\right) = \sum_{i=1}^{n} \operatorname{Var}\left(X_i\right).$$

Solution. Denote $\mathcal{F}_n = \sigma(X_1, \cdots, X_n)$. Using the law of total variance, we get

$$\begin{aligned} \operatorname{Var}\left(Z_n\right) &= \operatorname{Var}\left(E[Z_n|\mathcal{F}_{n-1}]\right) + E[\operatorname{Var}(Z_n|\mathcal{F}_{n-1})] = \operatorname{Var}(Z_{n-1}) + E[\operatorname{Var}(X_n|\mathcal{F}_{n-1})] \\ &= \operatorname{Var}(Z_{n-1}) + E[E[X_n^2|\mathcal{F}_{n-1}] - E[X_n|\mathcal{F}_{n-1}]^2] = \operatorname{Var}(Z_{n-1}) + E[X_n^2] \\ &= \operatorname{Var}(Z_{n-1}) + \operatorname{Var}(X_n) = \cdots = \sum_{i=1}^{n} \operatorname{Var}(X_i). \end{aligned}$$

Problem 3 (Ross Ex. 6.17)

Suppose that 100 balls are to be randomly distributed among 20 urns. Let X denote the number of urns that contain at least five balls. Use Azuma-Hoeffding inequality to derive an upper bound for $P\{X \geq 15\}$.
Solution. Let X_j denote the urn in which the j-th ball is placed for $j = 1, \cdots, 100$, then $X = h(X_1, \cdots, X_{100})$. It is straightforward to check that h satisfies the condition in Corollary 6.3.4. Hence,

$$P(X \geq 15) = P(X - E[X] \geq 15 - E[X]) \leq \exp\left(-(15 - E[X])^2/200\right),$$

where

$$E[X] = 20\left[1 - \sum_{k=0}^{4} \binom{100}{k} \left(\frac{1}{20}\right)^k \left(\frac{19}{20}\right)^{100-k}\right] \approx 11.28.$$

Problem 4 (Ross Ex. 6.22)

Let $\{X_n, n \geq 0\}$ be a Markov process for which X_0 is uniform on $(0,1)$ and, conditional on X_n,

$$X_{n+1} = \begin{cases} \alpha X_n + 1 - \alpha & \text{with probability } X_n \\ \alpha X_n & \text{with probability } 1 - X_n \end{cases}$$

where $0 < \alpha < 1$. Discuss the limiting properties of the sequence X_n, $n \geq 1$.
Solution. Note that $0 \leq X_n \leq 1$ for all n, and

$$E[X_{n+1}|X_1, \cdots, X_n] = E[X_{n+1}|X_n] = X_n(\alpha X_n + 1 - \alpha) + (1 - X_n)\alpha X_n = X_n.$$

Hence, $\{X_n\}$ is a bounded martingale. According to the martingale convergence theorem, $\{X_n\}$ converges almost surely to X where $P(X = 0) = P(X = 1) = 1/2$.