STATS 218 Homework 5 Solutions

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Let Z_n be the population size after n generations for a branching process with offspring distribution P_X. Namely, for each $n \geq 1$, let $(X_{n,k})_{k\geq 1}$ be i.i.d. random variables $X_{n,k} \sim P_X$ independent of $(X_{n',k})_{n'<n}$. For $n = 0$, set $Z_0 = 1$. For $n \geq 1$, set

$$Z_n = \sum_{k=1}^{Z_{n-1}} X_{n,k}.$$

Assume $\mu = \mathbb{E}[X_{n,k}] < \infty$, $v = \mathrm{Var}(X_{n,k}) < \infty$.

(1) Define $M_n := \mu^{-n} Z_n$. Prove that $M_n \overset{a.s.}{\to} M_\infty$ for a random variable M_∞.

(2) Consider the case $\mu < 1$. What does point (1) imply about $\lim_{n\to\infty} Z_n$? What is the distribution of M_∞ in this case?

(3) Consider next $\mu = 1$, and answer the same questions as in point (2).

(4) For $\mu > 1$, show that $\sup_n \mathbb{E}(M_n^2) \leq C < \infty$ for some constant C. What does this imply about $\mathbb{E}(M_\infty)$?

(5) Show that (always for $\mu > 1$) the previous points imply

$$\mathbb{P}\left(\lim_{n\to\infty} Z_n = \infty\right) > 0.$$

(the population diverges with positive probability)

(6) For $X \sim P_X$, define ($s \in \mathbb{R}_{>0}$)

$$G(s) := \mathbb{E}[s^X] = \sum_{k=0}^{\infty} \mathbb{P}(X = k) s^k.$$

Assume $\mu > 1$ and $\mathbb{P}(X = 0) > 0$. Show that there exists a unique $s_* \in (0, 1)$ such that $s_* = G(s_*)$.

(7) Define $Y_n = s_*^{Z_n}$. Show that Y_n is a martingale and prove that $Y_n \overset{a.s.}{\to} Y_\infty$ for some random variable $Y_\infty \in [0, 1]$.

(8) Argue that $Y_\infty \in \{0, 1\}$, i.e., Y_∞ cannot take values in $(0, 1)$ with positive probability.

(9) Deduce that

$$\mathbb{P}\left(\lim_{n\to\infty} Z_n = 0\right) = s_*.$$

Solution.

(1) Note that $\mathbb{E}[Z_n|Z_{n-1}] = \mu Z_{n-1}$, which implies that $\mathbb{E}[M_n|M_{n-1}] = M_{n-1}$. Therefore, (M_n) is a non-negative martingale. The conclusion follows from martingale convergence theorem.

(2) Point (1) implies $Z_n \overset{a.s.}{\to} 0$. Since Z_n is integer-valued, this means that almost surely, $Z_n = 0$ for large enough n. Hence, $M_n = 0$ for large enough n, and consequently, $M_\infty \equiv 0$.

(3) In this case $M_n = Z_n$, thus $\lim_{n\to\infty} Z_n = M_\infty$. If $X \equiv 1$, then of course $M_\infty \equiv 1$. Otherwise, we must have $p := \mathbb{P}(X = 0) > 0$. We show that $M_\infty = 0$ almost surely. For any $k \geq 1$,

$$\mathbb{P}(M_{n+1} = 0) \geq \mathbb{P}(M_n = 0) + \mathbb{P}(M_n = k)p^k.$$

Sending $n \to \infty$ implies $\mathbb{P}(M_\infty = k) = 0$. Hence, $M_\infty \equiv 0$.

(4) By direct computation,

$$\begin{aligned}
&\mathbb{E}[Z_n^2|Z_{n-1}] = \mathrm{Var}(X_{n,1})Z_{n-1} + \mathbb{E}[X_{n,1}]^2 Z_{n-1}^2 = \mu^2 Z_{n-1}^2 + vZ_{n-1}\\
\Rightarrow &\mathbb{E}[Z_n^2] = \mu^2\mathbb{E}[Z_{n-1}^2] + v\mathbb{E}[Z_{n-1}] = \mu^2\mathbb{E}[Z_{n-1}^2] + v\mu^{n-1}\\
\Rightarrow &\mathbb{E}[M_n^2] = \mu^{-2n}\mathbb{E}[Z_n^2] = \mu^{-2(n-1)}\mathbb{E}[Z_{n-1}^2] + \mu^{-(n+1)}v = \mathbb{E}[M_{n-1}^2] + \mu^{-(n+1)}v\\
\Rightarrow &\mathbb{E}[M_n^2] = 1 + v\sum_{k=1}^{n} \mu^{-(k+1)} \leq 1 + \frac{v\mu^{-2}}{1-\mu^{-1}} < \infty.
\end{aligned}$$

This further implies that the sequence (M_n) is uniformly integrable. Hence, $\mathbb{E}[M_\infty] = \mathbb{E}[M_0] = 1$.

(5) Since $\mathbb{E}[M_\infty] > 0$, we know that $\mathbb{P}(M_\infty > 0) > 0$. On this event, we have

$$\lim_{n\to\infty} M_n > 0 \Rightarrow \lim_{n\to\infty} Z_n = \infty.$$

(6) Only need to note that $G(0) > 0$, $G(1) = 1$ and $G'(1) > 1$, so $G(s) < s$ for s close to 1. Use intermediate zero theorem to prove existence. The uniqueness follows from the convexity of G.

(7) We have $\mathbb{E}[Y_n|Y_{n-1}] = \mathbb{E}[Y_n|Z_{n-1}]$. Given $Z_{n-1} = m$, we can calculate

$$\mathbb{E}[Y_n|Z_{n-1} = m] = \mathbb{E}\left[s_*^{Z_n}|Z_{n-1} = m\right] = \mathbb{E}\left[s_*^{\sum_{k=1}^m X_{n,k}}\right] = \mathbb{E}\left[s_*^{X_{n,1}}\right]^m = G(s_*)^m = s_*^m.$$

This implies $\mathbb{E}[Y_n|Y_{n-1}] = s_*^{Z_{n-1}} = Y_{n-1}$. Therefore, (Y_n) is a non-negative martingale and $Y_n \in (0,1)$. According to martingale convergence theorem, $Y_n \overset{a.s.}{\to} Y_\infty$ where $Y_\infty \in [0,1]$.

(8) Note that $Y_\infty \in (0,1) \Leftrightarrow Z_n \to Z_\infty \in (0,\infty)$ (since $s_* \in (0,1)$). We only need to show that for any $k \geq 1$, $\mathbb{P}(\lim_{n\to\infty} Z_n = k) = 0$. Since Z_n is integer-valued, we have

$$\mathbb{P}\left(\lim_{n\to\infty} Z_n = k\right) = \mathbb{P}\left(\exists N,\ \forall n \geq N,\ Z_n = k\right) \leq \sum_N \mathbb{P}\left(\forall n \geq N,\ Z_n = k\right).$$

For any fixed N, we have

$$\begin{aligned}
\mathbb{P}\left(\forall n \geq N,\ Z_n = k\right) =&\mathbb{P}\left(Z_N = k\right)\prod_{n=N}^{\infty}\mathbb{P}\left(Z_{n+1} = k|Z_n = k\right) \leq \mathbb{P}\left(Z_N = k\right)\prod_{n=N}^{\infty}\left(1 - \mathbb{P}\left(Z_{n+1} = 0|Z_n = k\right)\right)\\
\leq&\mathbb{P}\left(Z_N = k\right)\prod_{n=N}^{\infty}\left(1 - \mathbb{P}\left(X = 0\right)^k\right) = 0,
\end{aligned}$$

thus leading to $\mathbb{P}\left(\lim_{n\to\infty} Z_n = k\right) = 0$, as desired.

(9) We notice that $\mathbb{P}\left(\lim_{n\to\infty} Z_n = 0\right) = \mathbb{P}\left(\lim_{n\to\infty} Y_n = 1\right) = \mathbb{P}(Y_\infty = 1) = \mathbb{E}[Y_\infty] = \mathbb{E}[Y_0] = s_*$ (by bounded convergence theorem).