STATS 218 Homework 6 Solutions

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Problem 1 (Ross Ex. 8.1)

Let $Y(t) = tX(1/t)$.

(a) What is the distribution of $Y(t)$?

(b) Compute $\mathrm{Cov}(Y(s), Y(t))$.

(c) Argue that $\{Y(t), t \geq 0\}$ is also Brownian motion.

(d) Let

$$T = \inf\{t > 0 : \quad X(t) = 0\}.$$

Using (c) present an argument that

$$P\{T = 0\} = 1.$$

Solution.

(a) $N(0, t)$.

(b) $\min(s, t)$.

(c) By (b) and continuity.

(d) For any $\varepsilon > 0$, we have

$$\mathbb{P}(T \leq \varepsilon) = \mathbb{P}(\exists t \leq \varepsilon, X(t) = 0) = \mathbb{P}(\exists t \geq 1/\varepsilon, Y(t) = 0).$$

Note that for all $x \in \mathbb{R}$, $\mathbb{P}(\exists t \geq 1/\varepsilon, Y(t) = 0 | Y(1/\varepsilon) = x) = \mathbb{P}(\exists t \geq 0, Y(t) = 0 | Y(0) = x) = \mathbb{P}(\exists t \geq 0, Y(t) = -x | Y(0) = 0) = 1$ by recurrence. Therefore, $\mathbb{P}(T \leq \varepsilon) = 1$. Letting $\varepsilon \to 0^+$ gives $\mathbb{P}(T = 0) = 1$.

Problem 2 (Ross Ex. 8.2)

Let $W(t) = X\left(a^2 t\right)/a$ for $a > 0$. Verify that $W(t)$ is also Brownian motion.
Solution. Verify the two conditions.

Problem 3 (Ross Ex. 8.5)

A stochastic process $\{X(t), t \geq 0\}$ is said to be stationary if $X(t_1), \ldots, X(t_n)$ has the same joint distribution as $X(t_1 + a), \cdots, X(t_n + a)$ for all $n, a, t_1, \cdots, t_n$.

(a) Prove that a necessary and sufficient condition for a Gaussian process to be stationary is that $\mathrm{Cov}(X(s), X(t))$ depends only on $t - s$, $s \leq t$, and $E[X(t)] = c$.

(b) Let $\{X(t), t \geq 0\}$ be Brownian motion and define

$$V(t) = e^{-\alpha t/2} X\left(\alpha e^{\alpha t}\right)$$

Show that $\{V(t), t \geq 0\}$ is a stationary Gaussian process. It is called the Ornstein-Uhlenbeck process.

Solution.

(a) Gaussian process is uniquely characterized by mean and covariance function.

(b) By direct calculation, $\mathbb{E}[V(t)] = 0$, and

$\mathrm{Cov}(V(t), V(s)) = \exp(-\alpha(t+s)/2)\, \mathrm{Cov}(X\left(\alpha e^{\alpha t}\right), X\left(\alpha e^{\alpha s}\right)) = \exp(-\alpha(t+s)/2)\alpha \exp(\alpha \min(t, s)) = \alpha \exp(-\alpha|t-s|/2)$