STATS 218 Homework 7 Solutions

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Problem 1 (Ross Ex. 8.6)

Let $\{X(t), t \geq 0\}$ denote a birth and death process that is allowed to go negative and that has constant birth and death rates $\lambda_n \equiv \lambda, \mu_n \equiv \mu$, $n = 0, \pm 1, \pm 2, \cdots$. Define μ and c as functions of λ in such a way that $\{cX(t), t \geq u\}$ converges to Brownian motion as $\lambda \to \infty$. *Note:* The definition of birth and death process can be found in Section 5.3 of Ross.

Solution. To achieve zero mean, wo should choose $\mu = \lambda$. Then we know that the interarrival times are i.i.d. $\text{Exp}(2\lambda)$, with mean $1/2\lambda$. In the time interval $[0, t]$, there are approximately $2\lambda t$ birth/death's. Therefore, $\text{Var}(X(t)) \approx 2\lambda t$ for large λ. We should choose $c = 1/\sqrt{2\lambda}$ such that $\text{Var}(cX(t)) = 1$. One can show that $cX(t)$ weakly converges to $B(t)$ but it is not required.

Problem 2 (Ross Ex. 8.7)

Let $\{X(t), t \geq 0\}$ denote Brownian motion. Find the distribution of

(a) $|X(t)|$

(b) $|\min_{0 \leq s \leq t} X(s)|$

(c) $\max_{0 \leq s \leq t} X(s) - X(t)$.

Solution. We need the following:
Lemma. (reflection principle) $\mathbb{P}(\max_{0 \leq s \leq t} X(s) \geq a) = 2\mathbb{P}(X(t) \geq a) = \mathbb{P}(|X(t)| \geq a)$.

The random variables in (b) and (c) are equal in distribution to $\max_{0 \leq s \leq t} X(s)$ and $|X(t)|$.