STATS 218 Homework 8 Solutions

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Problem 1 (Grimmett Ex. 13.7.2)

Let W be a standard Wiener process. Fix $t > 0, n \geq 1$, and let $\delta = t/n$. Show that $Z_n = \sum_{j=0}^{n-1} \left(W_{(j+1)\delta} - W_{j\delta}\right)^2$ satisfies $Z_n \to t$ in mean square as $n \to \infty$.

Solution. Note that

$$Z_n - t = \sum_{j=0}^{n-1} \left(\left(W_{(j+1)\delta} - W_{j\delta}\right)^2 - \delta \right),$$

where for each j, $\mathbb{E}[(W_{(j+1)\delta} - W_{j\delta})^2] = (j+1)\delta - j\delta = \delta$, and they are mutually independent. Therefore,

$$\mathbb{E}\left[(Z_n - t)^2\right] = \operatorname{Var}\left(\sum_{j=0}^{n-1} (W_{(j+1)\delta} - W_{j\delta})^2 \right) = \sum_{j=0}^{n-1} \operatorname{Var}\left((W_{(j+1)\delta} - W_{j\delta})^2 \right).$$

Since $W_{(j+1)\delta} - W_{j\delta} \sim \mathsf{N}(0, \delta)$, we know that

$$\operatorname{Var}\left((W_{(j+1)\delta} - W_{j\delta})^2 \right) = \mathbb{E}\left[(W_{(j+1)\delta} - W_{j\delta})^4\right] - \mathbb{E}\left[(W_{(j+1)\delta} - W_{j\delta})^2\right]^2 = 3\delta^2 - \delta^2 = 2\delta^2,$$

since the fourth moment of $\mathsf{N}(0,1)$ is 3. This finally leads to

$$\mathbb{E}\left[(Z_n - t)^2\right] = 2n\delta^2 = \frac{2t^2}{n} \to 0 \textbf{ as } n \to \infty.$$

Problem 2 (Grimmett Ex. 13.7.3)

Let W be a standard Wiener process. Fix $t > 0, n \geq 1$, and let $\delta = t/n$. Let $V_j = W_{j\delta}$ and $\Delta_j = V_{j+1} - V_j$. Evaluate the limits of the following as $n \to \infty$:

(a) $I_1(n) = \sum_j V_j \Delta_j$,

(b) $I_2(n) = \sum_j V_{j+1} \Delta_j$,

(c) $I_3(n) = \sum_j \frac{1}{2}\left(V_{j+1} + V_j\right)\Delta_j$,

(d) $I_4(n) = \sum_j W_{\left(j+\frac{1}{2}\right)\delta} \Delta_j$.

Solution. In this problem, all limits are in L^2 sense.

(a) By definition, $\lim_{n\to\infty} I_1(n) = \int_0^t W_s \mathrm{d}W_s$.

(b) Note that $I_2(n) = I_1(n) + \sum_j \Delta_j^2$. According to Problem 1, $\sum_j \Delta_j^2 \to t$. Hence, $I_2(n) \to \int_0^t W_s \mathrm{d}W_s + t$.

(c) $I_3(n) = (I_1(n) + I_2(n))/2$. Therefore, $I_3(n) \to \int_0^t W_s \mathrm{d}W_s + t/2$.

(d) By direct calculation, we obtain that

$$I_4(n) - I_3(n) = \frac{1}{2}\sum_j \left(2W_{(j+1/2)\delta} - W_{j\delta} - W_{(j+1)\delta}\right)\Delta_j$$
$$= \frac{1}{2}\sum_j \left(\left(W_{(j+1/2)\delta} - W_{j\delta}\right) - \left(W_{(j+1)\delta} - W_{(j+1/2)\delta}\right)\right)\left(\left(W_{(j+1/2)\delta} - W_{j\delta}\right) + \left(W_{(j+1)\delta} - W_{(j+1/2)\delta}\right)\right)$$
$$= \frac{1}{2}\sum_j \left(\left(W_{(j+1/2)\delta} - W_{j\delta}\right)^2 - \left(W_{(j+1)\delta} - W_{(j+1/2)\delta}\right)^2\right)$$
$$= \frac{1}{2}\sum_j \left(W_{(j+1/2)\delta} - W_{j\delta}\right)^2 - \frac{1}{2}\sum_j \left(W_{(j+1)\delta} - W_{(j+1/2)\delta}\right)^2.$$

Similar to the proof of Problem 1, we can show that

$$\sum_j \left(W_{(j+1/2)\delta} - W_{j\delta}\right)^2 \to \frac{t}{2}, \quad \sum_j \left(W_{(j+1)\delta} - W_{(j+1/2)\delta}\right)^2 \to \frac{t}{2},$$

thus leading to $I_4(n) - I_3(n) \to 0$. Therefore, $I_4(n) \to \int_0^t W_s \mathrm{d}W_s + t/2$.

Problem 3 (Grimmett Ex. 13.12.7)

Let $X_0, X_1, \ldots$ be independent $N(0,1)$ variables, and show that

$$W(t) = \frac{t}{\sqrt{\pi}}X_0 + \sqrt{\frac{2}{\pi}}\sum_{k=1}^{\infty}\frac{\sin(kt)}{k}X_k$$

defines a standard Wiener process on $[0,\pi]$.
Solution. First we show that the series converges uniformly (along a subsequence), implying that the limit exists and is a continuous function of t. Set

$$Z_{mn}(t) = \sum_{k=m}^{n-1}\frac{\sin(kt)}{k}X_k, \quad M_{mn} = \sup\left\{|Z_{mn}(t)| : 0 \le t \le \pi\right\}. \qquad (*)$$

We have that

$$M_{mn}^2 \le \sup_{0\le t\le\pi}\left|\sum_{k=m}^{n-1}\frac{e^{ikt}}{k}X_k\right|^2 \le \sum_{k=m}^{n-1}\frac{X_k^2}{k^2} + 2\sum_{l=1}^{n-m-1}\left|\sum_{j=m}^{n-l-1}\frac{X_j X_{j+l}}{j(j+l)}\right|.$$

The mean value of the final term is, by the Cauchy-Schwarz inequality, no larger than

$$2\sum_{l=1}^{n-m-1}\sqrt{\mathbb{E}\left(\left|\sum_{j=m}^{n-l-1}\frac{X_j X_{j+l}}{j(j+l)}\right|^2\right)} = 2\sum_{l=1}^{n-m-1}\sqrt{\sum_{j=m}^{n-l-1}\frac{1}{j^2(j+l)^2}} \le 2(n-m)\sqrt{\frac{n-m}{m^4}}.$$

Combine this with $(*)$ to obtain

$$\mathbb{E}\left(M_{m,2m}\right)^2 \le \mathbb{E}\left(M_{m,2m}^2\right) \le \frac{3}{\sqrt{m}}.$$

It follows that

$$\mathbb{E}\left(\sum_{n=1}^{\infty} M_{2^{n-1},2^n}\right) \le \sum_{n=1}^{\infty} \frac{6}{2^{n/2}} < \infty,$$

implying that $\sum_{n=1}^{\infty} M_{2^{n-1},2^n} < \infty$ a.s. Therefore the series which defines W converges uniformly with probability 1 (along a subsequence), and hence W has (a.s.) continuous sample paths.

Certainly W is a Gaussian process since $W(t)$ is the sum of normal variables (see Problem (7.11.19)). Furthermore $\mathbb{E}(W(t)) = 0$, and

$$\operatorname{cov}(W(s), W(t)) = \frac{st}{\pi} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\sin(ks)\sin(kt)}{k^2},$$

since the X_i are independent with zero means and unit variances. It is an exercise in Fourier analysis to deduce that $\operatorname{cov}(W(s), W(t)) = \min\{s, t\}$.