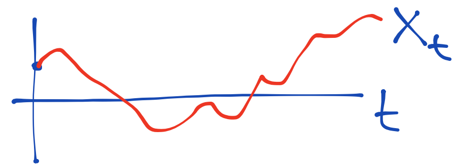
5/16/2022

Diffusion process $(X_t)_{t\geq 0}$

* $(X_t)_{t\geq 0}$ continuous sample paths
* Markov
* $\mathbb{E}\{(X_{t+h}-X_t) \mid X_t = x\} = a(t,x)h + o(h)$
* $\mathbb{E}\{(X_{t+h}-X_t)^2 \mid X_t = x\} = b(t,x)h + o(h)$

Uniquely identify the distribution of the sample path $(X_t)_{t\geq 0}$ (provided $X_0 \sim p_0$ is given)

Rmk $(X_t)_{t\geq 0}$ infinite dimensional.



its distr. determined by initializ + 2 functions a, b. □

Why? (1) Determine transition prob

(2) transition prob + Markov $\Rightarrow$ Whole distribution.

(1) Remember that, for $t \geq s$

$$F(t,y \mid s,x) = \mathbb{P}(X_t \leq y \mid X_s = x)$$

determined by

(i) <u>Initial condition</u>

$$\lim_{t \downarrow s} F(t, y \mid s, x) = 1_{y \geq x}.$$

(ii) <u>Forward diffusion eq</u> for $t > s$

$$f(t, y \mid s, x) = \frac{d}{dy} F(t, y \mid s, x)$$

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial y}\left(a(t, y) f\right) - \frac{1}{2} \frac{\partial^2}{\partial y^2}\left[b(t, y) f\right] = 0$$

If we know a, b can find unique solution of PDE s.t. initial condition

$\Rightarrow$ a, b determine uniquely f.

(2) <u>From transition prob to distr of</u> $(X_t)_{t \geq 0}$

Consider $t_0 = 0 < t_1 < \ldots < t_m$

For simplicity $X_0 \sim p_0$ where p_0 has density f_0

Joint distr of $(X_{t_0}, X_{t_1}, \ldots X_{t_m})$ has density $f^{(m)}_{t_0, \ldots t_m}(x_0, \ldots x_m)$ on $\mathbb{R}^{m+1}$

$$\boxed{f^{(m)}_{t_0, \ldots t_m}(x_0, \ldots x_m) = f_0(x_0) \prod_{i=1}^{m} f(t_i, x_i \mid t_{i-1}, x_{i-1})}$$

Follows by induction wrt m by using

$$f^{(m)}_{t_0,\dots t_m}(x_0,\dots x_m) = f^{(m-1)}_{t_0 \dots t_{m-1}}(x_0,\dots x_{m-1})\, f(t_m, x_m \mid t_{m-1}, x_{m-1}) \leftarrow$$

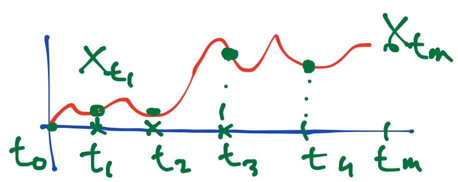
(Markov property

$$\mathbb{P}(X_{t_m} \le x_m \mid X_{t_0} = x_0 \dots X_{t_{m-1}} = x_{m-1})$$

$$= \mathbb{P}(X_{t_m} \le x_m \mid X_{t_{m-1}} = x_{m-1}) = F(t_m, x_m \mid t_{m-1}, x_{m-1})$$

Transition prob. uniquely determine joint distr of $(X_{t_0}, X_{t_1} \dots X_{t_m})$.

$\Rightarrow$ $a(\cdot), b(\cdot)$ uniquely determine $\mathcal{J}$



$\Rightarrow$ uniquely determine distr of $(X_t)_{t \ge 0}$

Intuition: Take a finer and finer mesh $(t_0, \dots t_m)$

$t_i = i/m \qquad 1 \le i \le m$

Since $t \mapsto X_t$ continuous.