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Ornstein-Uhlenbeck process

$$\mathbb{E}\{(X_{t+h}-X_t) \mid X_t = x\} = a(t,x)h + o(h)$$

$$\mathbb{E}\{(X_{t+h}-X_t)^2 \mid X_t = x\} = b(t,x)h + o(h)$$

$a(t,x) = a_0 \qquad b(t,x) = b_0 \quad \Rightarrow$ BM with drift

$$a(t,x) = a_0 + a_1 x \qquad b(t,x) = b_0$$

$$= a_1\left(\frac{a_0}{a_1} + x\right)$$

Can always reduce to $(t' = ct, \; x' = c_0 + c_1 x)$

$$a(t,x) = \pm x \qquad b(t,x) = 2$$

OU process corresponds to $a(t,x) = -x$

$$\mathbb{E}\{(X_{t+h}-X_t) \mid X_t = x\} = -xh + o(h)$$

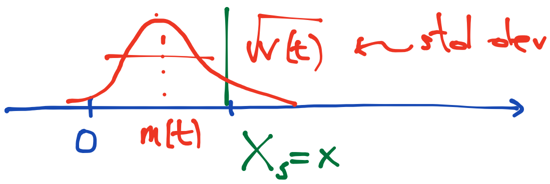
$$\mathbb{E}\{X_{t+h} \mid X_t = x\} = \underbrace{(1-h)}_{\text{converge towards } 0} x + o(h)$$

Forward Diffusion Equation

$$\frac{\partial f}{\partial t} - \frac{\partial}{\partial y}(\underbrace{y}_{a} f) - \underbrace{\frac{\partial^2 f}{\partial y^2}}_{\frac{1}{2}b} = 0$$

$$f(t,y \mid s,x) = \frac{1}{\sqrt{2\pi v(t)}} \exp\left\{ -\frac{(y-m(t))^2}{2v(t)} \right\} \qquad (*)$$

$$m(t) = m(t \mid s,x)$$
$$v(t) = v(t \mid s,x)$$
omit these dependencies



$$\lim_{t \downarrow s} F(t,y \mid s,x) = 1_{y \geq s}$$

$$F(t,y \mid s,x) = \Phi\left(\frac{y - m(t)}{\sqrt{v(t)}} \right)$$

$$\lim_{t \downarrow s} v(t \mid s,x) = 0 \; ; \qquad \lim_{t \downarrow s} m(t) = x \, .$$

$$\frac{\partial f}{\partial t} = -\frac{v'}{2v} f + \frac{(y-m)}{v} m' f + \frac{(y-m)}{2v^2} v' f$$

$$\frac{\partial (yf)}{\partial y} = f - y \frac{(y-m)}{v} f$$

$$\frac{\partial^2 f}{\partial y^2} = -\frac{1}{v} f + \frac{(y-m)^2}{v^2} f$$

Substituting in FDE: This is a sol if and only if

$$\begin{cases} m'(t) = -m(t) \\ v'(t) = 2\,(1 - v(t)) \end{cases}$$

$$\begin{cases} m(t) = m(s)\, e^{-(t-s)} \\ v(t) = 1 - (1 - v(s))\, e^{-2(t-s)} \end{cases}$$

$$m(s) = x \, , \quad v(s) = 0$$

$$\begin{cases} m(t) = x e^{-(t-s)} & \text{depends linearly in } x \\ v(t) = 1 - e^{-2(t-s)} =: v(t \mid s) & \text{does not depend on } x \end{cases}$$

$$f(t,y\,|\,s,x) = \frac{1}{\sqrt{2\pi v(t|s)}} \exp\left\{ - \frac{\left(y - x e^{-(t-s)}\right)^2}{2 v(t|s)} \right\} \qquad (*)$$

Conditional on $X_s = x$

$$X_t = x e^{-(t-s)} + \sqrt{v(t|s)}\, G \quad , \; G \sim N(0,1)$$

Equivalently

$$\begin{cases} X_t = X_s e^{-(t-s)} + \sqrt{1 - e^{-2(t-s)}}\; G & (*) \\ G \sim N(0,1) \text{ indep of } X_s \end{cases}$$

<u>Rmk 1</u> As $t \to \infty$

$$\lim_{t\to\infty} f(t,y\,|\,s,x) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2} =: f_{eq}(y)$$

<u>Rmk2</u> If $X_s \sim N(\mu_s, \sigma_s^2)$ then (X_s, X_t) jointly Gaussian.

<u>Consequence of general fact</u>

$Z \sim N(\mu, \Sigma)$ in $\mathbb{R}^d$

$$f_{X|Z}(x|z) = \frac{1}{\sqrt{2\pi\alpha^2}} e^{-\frac{(x - c_0 - b^T z)^2}{2\alpha^2}}$$

then (Z, X) is Gaussian.

$$f_{Z,X}(z,x) \propto \exp\left\{ -\frac{1}{2}\langle z-\mu, \Sigma^{-1}(z-\mu)\rangle - \frac{1}{2\alpha^2}(x - c_0 - b^T z)^2 \right\}$$

Rmk3 If $X_{t_0} \sim N(\mu_0, \sigma_0^2)$ $t_0 = 0$. then

$f^{(m)}_{t_0,\dots t_m}$ joint distr of $(X_{t_0}, \dots X_{t_m})$

is a multivariate Gaussian.

[I.e. $(X_t)_{t \geq 0}$ is a Gaussian process]

Further if

$$X_s \sim \underset{N(0,1)}{f_{eq}} \quad \text{then } \forall t > s \quad X_t \sim f_{eq}$$

Easiest way to see it

$$X_t = e^{-(t-s)} X_s + \sqrt{1 - e^{-2(t-s)}}\, G$$

$G \sim N(0,1)$ indep of X_s

- X_t = sum of indep Gaussian

$$\Rightarrow X_t \sim N(\mu, \sigma^2)$$

$\mu = 0 \qquad \sigma^2 = 1$

Alternative approach to show this

$$\rightarrow * \quad f_t(y) = \int_{\mathbb{R}} f(t, y \mid s, x)\, f_{eq}(x)\, dx$$

* Compute the integral to check

$$f_t(y) = f_{eq}(y)$$

Conseq.

If $X_0 \sim N(0,1)$ then

$$\underline{f_{t_1 \ldots t_m}(x_1 \ldots x_m)} = f_{eq}(x_1) \prod_{i=2}^{m} \underbrace{f(t_i, x_i \mid t_{i-1}, x_{i-1})}_{\text{depends only on } t_i - t_{i-1}}$$

$$= f_{t_1+c, \ldots, t_m+c}(x_1 \ldots x_m) \quad \text{for any shift } c .$$