Stat 218 3/28/2022

Logistics

Books: Grimmett Stirzaker (Ch 10, 12, 13)
Ross (Ch 3, 6, 8)
Karlin. Taylor (Ch 5, 6, 7)

- Renewal Processes
- Martingales and Applications
- Brownian motion and diffursions

1 hw/week, 1 midterm, 1 final.

Prereq: • Analysis, Basic probability. (not meas th)
• 217 Material helpful not required

Today A refresher about convergence of r.v's.

$\Omega = \{T,H\}^{\mathbb{N}}$

Ω sample space (e.g. $\Omega = [0,1]$ }, $\Omega = \mathbb{R}$

$\mathbb{P}$ prob distr over Ω $A \mapsto \mathbb{P}(A)$, $A \subseteq \Omega$

X r.v $X : \Omega \to \mathbb{R}$, $\omega \mapsto X_n(\omega)$.

$(X_n)_{n \in \mathbb{N}}$ seq of r.v., X_∞

Notions of convergence

- $X_n \xrightarrow{a.s.} X_\infty$
- $X_n \xrightarrow{P} X_\infty$
- $X_n \xrightarrow{L^q} X_\infty$
- $X_n \overset{d}{\Longrightarrow} X_\infty$

} these are statements about the r.v. themselves

} this is about the distr

- Give definitions.

$$X_n \xRightarrow{d} X_\infty \qquad F_n(t) := \mathbb{P}(X_n \leq t)$$

$$\lim_{n\to\infty} \mathbb{E}\, h(X_n) = \mathbb{E}\, h(X_\infty) \quad \forall\ h \text{ bdd ct.}$$

$$\lim_{n\to\infty} F_n(t) = F_\infty(t) \quad \forall\ t \text{ cont point of } F_\infty.$$

– Classical example of a.s. conv.

$$X_n = \frac{1}{n}\sum_{i=1}^{n} Z_i \qquad (Z_i)_{i\geq 1} \text{ iid}$$

$$\mathbb{E}|Z_i| < \infty$$

SLLN

$$X_n \xrightarrow{a.s.} \mu \qquad \mu := \mathbb{E}X_1.$$

– Classical example of $\xRightarrow{d}$

$$X_n = \frac{1}{\sqrt{n\sigma^2}}\sum_{i=1}^{n} (Z_i - \mu) \qquad Z_i \text{ iid}$$

$$\mathbb{E}Z_i = \mu, \quad \mathbb{E}[(Z_i-\mu)^2] = \sigma^2 < \infty$$

CLT

$$\frac{1}{\sqrt{n}} X_n \xrightarrow{d} N(0,1)$$

(Some) Relations

$$(X_n \xrightarrow{a.s.} X_\infty) \Rightarrow (X_n \xrightarrow{P} X_\infty) \Rightarrow (X_n \xrightarrow{d} X_\infty)$$

example $X_n \xrightarrow{d} X_\infty$ but $X_n \not\xrightarrow{P} X_\infty$.

$\Omega = \{0,1\}$ $\quad \mathbb{P} = \text{unif.}$

$$X_n(\omega) = \begin{cases} 1_{\{\omega=0\}} & n \text{ even} \\ 1_{\{\omega=1\}} & n \text{ odd.} \end{cases}$$

$\mathbb{E}h(X_n) = \frac{1}{2}h(0) + \frac{1}{2}h(1)$ indep of n

$X_\infty = X_{even} \cdot 1_{\omega=0}$.

example $X_n \xrightarrow{P} X_\infty$ but $\cancel{X_n \xrightarrow{as} X_\infty}$

$X_1(\omega) = 1.$

$X_2(\omega) = 1_{\omega \in [0, 1/2]}$ $X_3(\omega) = 1_{[\frac{1}{2}, 1] \ni \omega}$

$X_4(\omega) = 1_{\omega \in [0, 1/3]}$ $X_5(\omega) = 1_{[1/3, 2/3]}$ $X_6 = 1_{[\frac{2}{3}, 1]}$

$X_7 = 1_{\omega \in [0, 1/4]}, \ldots$

$\boxed{Z}$