Brownian motion.

Consider a particle moving ~~on~~ on the real line $\mathbb{R}$.

Position at time $t \geq 0$ $X(t)$

It moves in discrete steps

$t \in \mathbb{N}\tau = \{0, \tau, 2\tau, \dots\}$

$$X_\tau((k+1)\tau) = X_\tau(k\tau) + \delta \cdot U_k$$

where $(U_k)_{k \geq 1}$ is a sequence of iid rv $\mathbb{E} U_k = 0$, $\mathbb{E} U_k^2 = 1$.

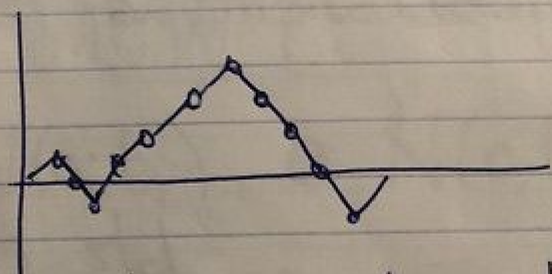
We interpolate linearly to get a random function

$$t \mapsto X_\delta(t)$$

Linear interpolation

For $t \in [k\tau, (k+1)\tau]$

$$X_\delta(t) := \left(\frac{(k+1)\tau - t}{\tau}\right) X(k\tau) + \left(\frac{t-k\tau}{\tau}\right) X((k+1)\tau)$$



What does this look like when $\delta, \tau \to 0$

Fix $t > 0$

$$X_\delta(t) = \delta \sum_{k=1}^{\lfloor t/\tau \rfloor} U_k + \underbrace{\delta \cdot \left(\frac{t - \lfloor \frac{t}{\tau} \rfloor \cdot \tau}{\tau}\right) \cdot U_{\lfloor \frac{t}{\tau} \rfloor + 1}}_{O(\delta)}$$

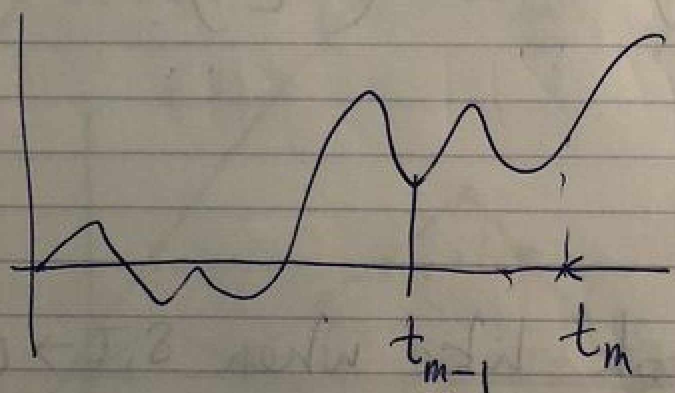
$$\frac{1}{\delta \lfloor \frac{t}{\tau} \rfloor^{1/2}} X_\delta(t) \xrightarrow{d} N(0,1)$$ as $\tau \to 0$ any δ

$$\frac{\tau^{1/2}}{\delta t} X_\delta(t) \xrightarrow{d} N(0, t)$$

Need to take $\boxed{\delta = \sqrt{\tau}}$

What about m distinct times?

$0 < t_1 < t_2 < \ldots < t_m$



Condition on $\left(X_\delta(t)\right)_{t \leq \lceil \frac{t_{m-1}}{\tau} \rceil \tau}$

$$\lim_{\delta \to 0} \mathbb{P}\left(X_\delta(t_m) - X_\delta(t_{m-1}) \leq s \,\middle|\, \left(X_\delta(t)\right)_{t \leq \lceil \frac{t_{m-1}}{\tau} \rceil}\right)$$

$$= \mathbb{P}\left(N(0, t_m - t_{m-1}) \leq s\right)$$

What is the joint distr?

$(X_\delta(t_1), X_\delta(t_2) - X_\delta(t_1), \ldots X_\delta(t_m) - X_\delta(t_{m-}$

are asymptotically indep. Gaussian

$N(0, t_1), \otimes N(0, t_2 - t_1) \otimes \ldots \otimes N(0,$

A std Brownian Motion $(X(t))_{t \geq 0}$ (Wiener process) is a random fct. satisfying 2 conditions

(1) $(X(t))_{t \geq 0}$ is continuous w prob $= 1$

(2) $\forall 0 \leq t_1 < t_2 \ldots < t_m$

$(X(t_1), X(t_2) - X(t_1) \ldots \quad X(t_m) -$

is a vector with indep Gaussian e

$$\mathbb{E}[(X(t_i)-X(t_{i-1}))^2] = t_i - t_{i-1}.$$

Equivalently $\forall t \geq t_0$

(1') $$\mathbb{P}(X(t) \leq s \mid (X(u))_{u \leq t_0}) = \mathbb{P}(N(0,t-t_0) \leq s)$$

It is a nontrivial theorem that this ~~d~~ prob distr over function can be constructed and is actually completely determined by the above 2 properties.

If $(W_t)_{t \geq 0}$ is a std BM.

* BM started at a: $X_t = a + W_t$
* BM started at a, with drift v

$$X_t = a + vt + W_t$$

<u>Definition</u> Given a continuous time process $(X_t)_{t\geq 0}$, of cont samplepaths T is a stopping time with respect to (X_t) if $\forall t \in \mathbb{R}_{\geq 0}$

$$\mathbb{1}_{T\leq t} = F((X_s)_{s\leq t}).$$

<u>Def</u> Y_t is a MG wrt $(X_t)_{t\geq 0}$ if $\forall t$ $\mathbb{E}|Y_t| < \infty$ and

$$\mathbb{E}(Y_{t+h} | (X_s)_{s\leq t}) = Y_t .$$

<u>Example</u> $(W_t)_{t\geq 0}$ BM started at $a > 0$

$$T = \inf\{t \geq 0 : W_t \in \{0, A\}\}$$

Hitting time at $0, A$.

$Y_t = e^{\theta W_t - \frac{\theta^2 t}{2}}$ is a MG wrt (W_t)

$\mathbb{E}\{Y_{t+h} \mid (W_s)_{s\leq t}\} = \mathbb{E}\{e^{\theta(W_{t+h} - W_t) - \frac{\theta^2}{2}h} Y_t \mid \{W_{\leq t}$

$= Y_t \mathbb{E}[e^{\theta(W_{t+h} - W_t)} \mid W_{\leq t}] e^{-\frac{h\theta^2}{2}} = Y_t$

recall that

$W_{t+h} - W_t$ is indep. of $W_{\leq t}$

$\sim N(0,h)$ [true also if you start at a, since $W_t = W_t^0 + a$ ...
started at 0]

it can be proven

$T < \infty$ a.s $Y_{t \wedge T}$ is a MG as well

$\mathbb{E}[Y_T] = \mathbb{E}(Y_0) = e^{\theta a}$

$\|$

$\mathbb{E}[e^{\theta A - \frac{\theta^2 T}{2}} 1_{T_A < T_b} + e^{-\frac{\theta^2 T}{2}} 1_{T_b < T_A}]$

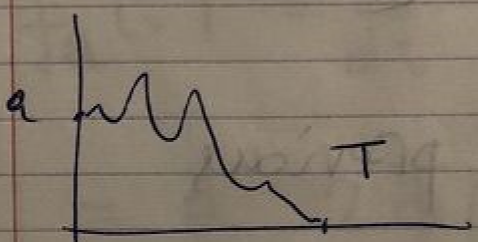
$$e^{\theta a} = \mathbb{E}\left[\left(e^{\theta A} 1_{T_A<T_0} + 1_{T_0<T_A}\right) e^{-\frac{\theta^2}{2}T}\right]$$

$$e^{-\theta a} = \mathbb{E}\left[\left(e^{-\theta A} 1_{T_A<T_0} + 1_{T_0<T_A}\right) e^{-\frac{\theta^2 T}{2}}\right]$$

$$e^{\theta a} + e^{\theta(A-a)} = \left(1+e^{\theta A}\right) \mathbb{E}\left[e^{-\theta^2 T/2}\right]$$

$$\mathbb{E}\left[e^{-\frac{\theta^2 T}{2}}\right] = \frac{e^{\theta a} + e^{\theta(A-a)}}{1+e^{\theta A}}$$

& take limit $A \to \infty$



$T = T_0$

$$\mathbb{E}\left[e^{-\frac{\theta^2}{2}T}\right] = e^{-\theta a}$$

$$\mathbb{E}\left[e^{-\lambda T}\right] = e^{-a\sqrt{2\lambda}}$$

$f_a(t)$ density of T

$$\int_0^\infty e^{-\lambda t} f_a(t)\, dt = e^{-a\sqrt{2\lambda}}$$