4/4/2022

Recall

$$\frac{N(t)}{t} \xrightarrow{a.s} \mu.$$

$N(t) \approx \mu t$ what are deviation

<u>Thm</u> (CLT) Assume $\mathbb{E}(X_1) = \mu < \infty$

$Var(X_1) = \sigma^2 < \infty$. Then

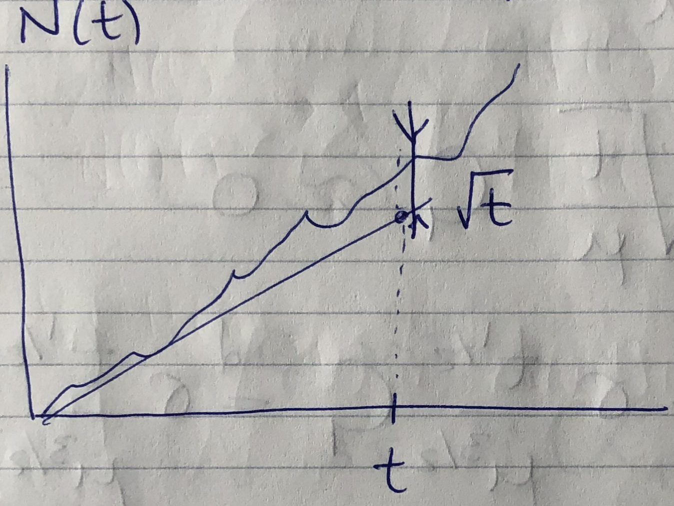
$$\frac{N(t) - t/\mu}{\sigma \mu^{-3/2} t^{1/2}} \overset{d}{\Longrightarrow} G \sim N(0,1).$$

<u>Recall</u> $\quad Z(t) \overset{d}{\Longrightarrow} G \sim N(0,1)$

$\mathbb{E}h(Z(t)) \to \mathbb{E}h(G).$ or.

Roughly speaking

$$N(t) \approx N\left(\frac{t}{\mu}, \frac{\sigma^2}{\mu^2} t\right)$$



<u>Heuristic</u>

$$T_{N(t)} \approx t$$

$$T_n = \sum_{i=1}^{n} X_i = \mu n + \sigma\sqrt{n}\, G_n$$

$$G_n \approx G \sim N(0,1)$$

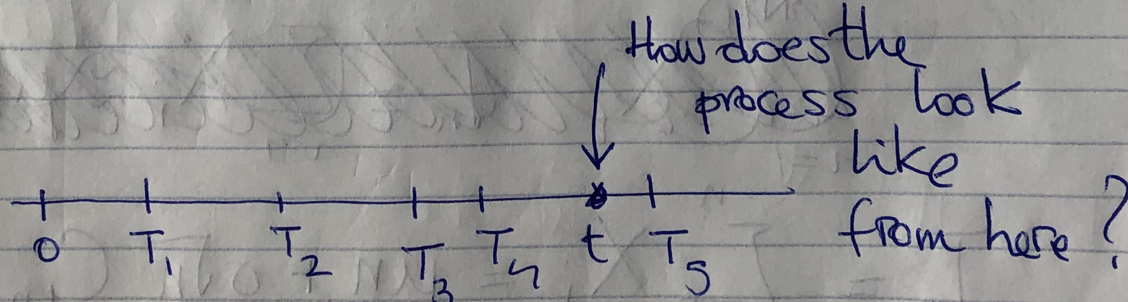
$$\mu N(t) + \sigma\sqrt{N(t)}\, G \approx t$$

$$N(t) = \frac{t}{\mu} + \Delta(t)$$

$$\mu\,\Delta(t) + \sigma\sqrt{\frac{t}{\mu}}\, G \approx 0$$

$$\Delta(t) \approx -\sigma \frac{t^{1/2}}{\mu^{3/2}} G \overset{d}{=} \sigma \frac{t^{1/2}}{\mu^{3/2}} G.$$

~~Renewal~~ We are often interested in



Eg: What is the at number of arrivals in the last 5 minutes

$$g(s) = \begin{cases} 1 & \text{if } 0 \leq s \leq \Delta \\ 0 & \text{ow} \end{cases}$$

$$\sum_{i=1}^{\infty} g(t - T_i) = \int_0^t g(t-x)\, dN(t)$$

The expectation is $\int_0^t g(t-x)\, dm(t)$

$m(t) = \mathbb{E} N(t)$

Assume X_1 non-arithmetic

$(\nexists\ \delta \in \mathbb{R}_{\geq 0}$ st $\mathbb{P}(X_1 \in \underbrace{\delta \mathbb{N}}_{\{0, \delta, 2\delta, \dots\}}) = 1 \quad)$

<u>Theorem</u> (Key renewal thm)

- X_1 non arithm. $\mathbb{E} X_1 = \mu < \infty$
- g non incr, $\int_0^{\infty} g(t)\, dt < \infty \quad g \geq 0$

Then

$$\lim_{t \to \infty} \int_0^t g(t-x)\, dm(\cancel{t}) = \frac{1}{\mu} \int_0^{\infty} g(s)\, ds$$

Turns out to be equivalent to a seemingly simpler result

<u>Thm</u> (Blackwell renewal thm)

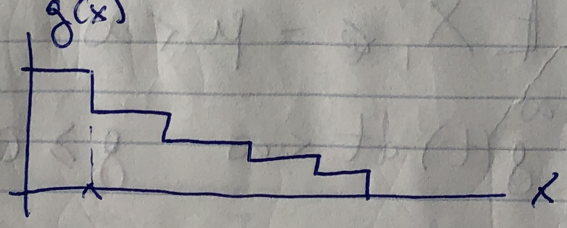
- X_1 non arithm. $\mu = \mathbb{E} X_1 < \infty$

Then $\forall h > 0$

$$\lim_{n \to \infty} [m(t+h) - m(t)] = \frac{h}{\mu} \qquad (*)$$

Indeed this implies KRT for

$$g(x) = \sum_{i=1}^{n} c_i \, 1_{x \in [0, x_i]}$$



~~BR~~ <u>Intuition</u> $N(t) \approx t/\mu$ for large t

We expect $m(t) \approx t/\mu$

BRT saysomething sharper

$\frac{dm}{dt} \approx 1/\mu$ for large t

We will prove the simpler result

<u>Thm</u> (Elementary renewal theorem)

– ~~X_1 non arithmetic~~ $\mu = \mathbb{E}X_1 < \infty$

$\Rightarrow \lim_{t\to\infty} \frac{m(t)}{t} = 1/\mu$ $\square$

Proof is ~~base d~~ based on a result of indep. interest

An integer-valued rv M is called a stopping time for $(X_i)_{i\geq 1}$ if $\forall\, m \in \mathbb{N}$

$$I(M \leq m) = f_m(X_1, \ldots X_m)$$

Informally
At any time m, I can figure out whether ~~...~~ time M already passed by looking at data up to that time

Lemma (Wald) $\mathbb{E}|X_i| < \infty$ $(X_i)_{i \geq 1}$ iid
M a stopping time $\mathbb{E}M < \infty$
Then

$$\mathbb{E}\left\{\sum_{i=1}^{M} X_i\right\} = \mathbb{E}(M)\,\mathbb{E}(X_1)$$

Proof (Wald)

$$\sum_{i=1}^{M} X_i = \sum_{i=1}^{\infty} X_i 1_{M \geq i}$$

$$\mathbb{E}\left(\sum_{i=1}^{M} X_i\right) = \mathbb{E}\left(\sum_{i=1}^{\infty} X_i 1_{M \geq i}\right) \overset{DCT}{=}$$

$$\leq \sum_{i=1}^{\infty} |X_i| 1_{M \geq i}$$

$$F(X_1,.. X_{i-1})$$

$$= \sum_{i=1}^{\infty} \mathbb{E}(X_i \mathbb{1}_{M \geq i}) \overset{!}{=} \sum_{i=1}^{\infty} \mathbb{E}(X_i) \mathbb{P}(M \geq i)$$

$$= \mathbb{E}(X_1) \sum_{i=1}^{\infty} \mathbb{P}(M \geq i) = \mathbb{E}(X_1) \mathbb{E}(M)$$

Example

(1) X_i iid $\text{Bern}(p)$ $\quad X_i \in \{0,1\}$

$$N_k = \min\{n : \sum_{i=1}^{n} X_i = k\}$$

$k = \mathbb{E}N_k \cdot p$

(2) $\mathbb{P}(X_i = +1) = p = 1 - \mathbb{P}(X_i = -1)$ $\quad \mathbb{E}N_k = \frac{k}{p}$

in (1) $N_k^+ = \max\{n : \sum_{i=1}^{n} X_i = k\}$

not a stopping time $\quad N_k^+ = N_{k+1} - 1$

$$\mathbb{E}N_k^+ = \cancel{\mathbb{E}(N)} + \frac{k+1}{p} - 1$$

$$= \frac{k}{p} + \underbrace{\sum_{\ell=0}^{\infty} (1-p)^{\ell} p \ell}_{\frac{1}{p} - 1}$$