4/6/2022

Proof of Elementary Renewal Theorem

$\left[\mu := \mathbb{E} X_1 < \infty \Rightarrow \lim_{t \to \infty} \frac{m(t)}{t} = \frac{1}{\mu} \right]$:

We use $T_{N(t)} \leq t \leq T_{N(t)+1}$

$$\sum_{i=1}^{N(t)} X_i \leq t \leq \sum_{i=1}^{N(t)+1} X_i$$

Note that $N(t)+1$ is a stopping time:

$$\{N(t)+1 \geq m\} = \{N(t) \geq m\} = \{\sum_{i=1}^{m} X_i \leq t\}$$
$$= F_m(X_1, \dots X_m)$$

By Wald

$$t \leq \mathbb{E}(N(t)+1)\, \mathbb{E} X_1 = [m(t)+1]\mu$$

$$\frac{m(t)}{t} \geq \frac{1}{\mu} - \frac{1}{t} \to \frac{1}{\mu} -$$

We need a u.b.: Try to use the firs ineq.

Note that $N(t)$ is not a s.t.!

Can try

$$t \geq \mathbb{E}\left\{\sum_{i=1}^{N(t)+1} X_i - X_{N(t)+1}\right\} =$$

$$= \mu\,[m(t)+1] - \underbrace{\mathbb{E}X_{N(t)+1}}_{\text{such} \neq \mu!} \quad (*)$$

→ suff to show bdd!

Truncation: Fix large constant a

$$X_i^a := \begin{cases} X_i & \text{if } X_i \leq a \\ a & \text{ow} \end{cases}$$

$N^a(t)$, $m^a(t)$, μ^a

Clearly $N(t) \leq N^a(t)$

$m(t) \leq m^a(t)$,

By (*)

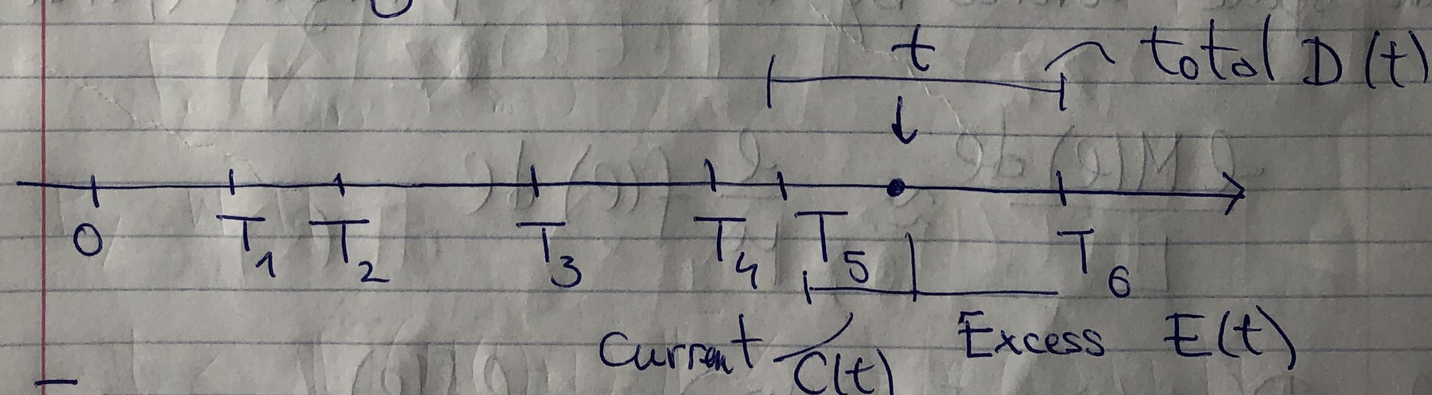
$$t \cancel{x} = \mu^a\left[m^a(t)+1\right] - \mathbb{E}X^a_{N(t)+1}$$
$$\geq \mu^a\left[m^a(t)+1\right] - a$$

$$\frac{m(t)}{t} \leq \frac{1}{\mu^a} - \frac{1}{t} + \frac{a}{\mu^a t}$$

$$\lim_{t\to\infty} \frac{m(t)}{t} \leq \frac{1}{\mu^a}$$

Take $a \to 0$ + monotone convergence.

Let us go back to this:

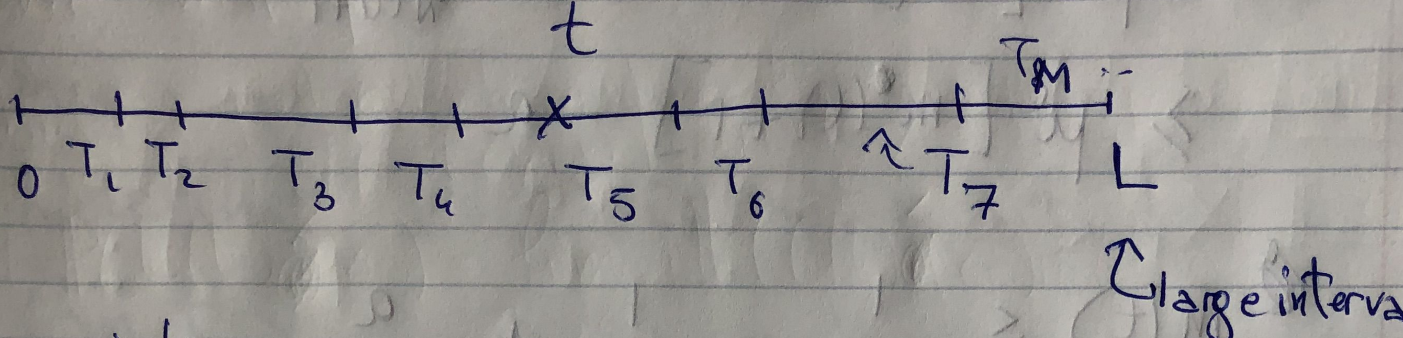


$E(t) := T_{N(t)+1} - t$

$C(t) := t - T_{N(t)}$

$D(t) := T_{N(t)+1} - T_{N(t)} = E(t) + D(t)$

Heuristic (assuming T has density f)



~~# int~~

$M \approx \frac{L}{\mu}$ # intervals

~~$M_t\, dt$~~ # intervals of length $(\ell, \ell+d\ell)$

$M(\ell)\, d\ell \approx \frac{L}{\mu} f(\ell)\, d\ell$

Frac. space covered by intervals of length $(\ell, \ell+d\ell)$

$$\frac{\ell M(\ell)\, d\ell}{L} \approx \frac{\ell}{\mu} f(\ell)\, d\ell$$

$$\lim_{t\to\infty} P(D(t) \in [\ell, \ell+\delta)) = \frac{\ell f(\ell)}{\mu}\delta + o(\delta)$$

$$\lim_{t\to\infty} \mathbb{P}(C(t)\geq c, E(t)\geq e) = \cancel{\int_{c+e}^{\infty} \frac{f(e)}{\mu}}$$

$$= \int_{c+e}^{\infty} \frac{x f(x)}{\mu}\left(1 - \frac{c+e}{x}\right) dx$$

$$= \frac{1}{\mu}\int_{c+e}^{\infty} f(x)(x-c-e)\, dx$$

In particular

$$\lim_{t\to\infty} \mathbb{P}(C(t)\geq z) = \frac{1}{\mu}\int_{z}^{\infty} f(x)(x-z)\, dx \quad (*)$$

$$= \lim_{t\to\infty} \mathbb{P}(E(t)\geq z)$$

Integr. by parts

$$\cancel{(*) = \frac{1}{\mu}\left[F(x)(x-z)\right]_{z}^{\infty} - \frac{1}{\mu}\int_{z}^{\infty}\left[F(x)\right]dx}$$

$$\cancel{= \frac{1}{\mu}\int_{0}}$$

We will prove

Thm

$$\lim_{t\to\infty} \mathbb{P}(E(t) > z) = \lim_{t\to\infty} \mathbb{P}(C(t) > z) =$$

$$= \frac{1}{\mu} \int_z^{\infty} (x-z)\, dF(x)$$

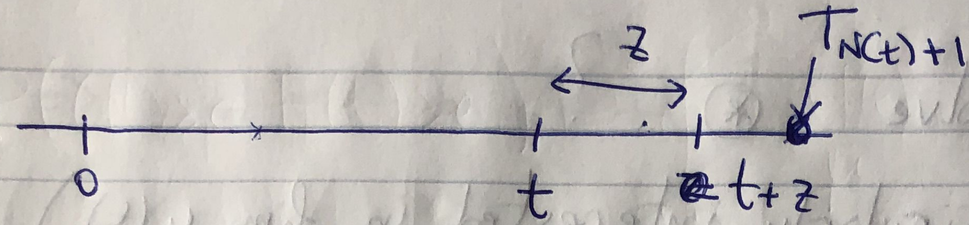
$$= 1 - \frac{1}{\mu} \int_0^{z} (1-F(x))\, dx \qquad \square$$

Proof

$$\mathbb{P}(E(t) > z) = \mathbb{E}\, \mathbb{P}(E(t) > z \mid X_1) =$$

$$= \int_0^{\infty} \mathbb{P}(E(t) > z \mid X_1 = x)\, dF(x)$$

$$= \cancel{\int_0^{z} \mathbb{P}(E(t) > z \mid X_1 = x)\, dF(x) + \int_z^{\infty}}$$



$$= \int_0^t \mathbb{P}(E(t) > z \mid X_1 = x)\, dF(x) +$$

$$+ \int_t^{t+z} \underbrace{\mathbb{P}(E(t) > z \mid X_1 = x)}_{=0}\, dF(x) +$$

$$+ \int_{t+z}^{\infty} \underbrace{\mathbb{P}(E(t) > z \mid X_1 = x)}_{=1}\, dF(x)$$

For $x \in [0, t]$

$$\mathbb{P}(E(t) > z \mid X_1 = x) = \mathbb{P}(E(t-x) > z)$$

$$\mu(t) := \mathbb{P}(E(t) > z)$$

$$\mu(t) = \int_0^t \mu(t-x)\, dF(x) + \underbrace{1 - F(t+z)}_{H(t)}$$

$$\mu(t) = H(t) + \int_0^{\infty} \mu(t-x)\, dF(x) \qquad (*)$$

$\circledast \quad \mu = H + F * \mu$

How do we solve $\circledast$?

(we are particularly interested in $\lim_{t\to\infty} \mu(t)$)

Let us derive a similar eq for $m(t)$

$$m(t) = \mathbb{E}N(t) = \int_0^t \mathbb{E}[N(t) \mid X_1 = x]\, dF(x)$$

$$= \int_0^t \mathbb{E}[N(t-x)+1]\, dF(x) =$$

$$= F(t) + \int_0^t m(t-x)\, dF(x)$$

$$\boxed{m = F + F * m} \quad \text{(renewal eq)}$$

$$m = F + F*F + F*F*F + \dots$$

$$= \sum_{l=1}^{\infty} F^{*l}$$

$$\mu = H + F*H + F*F*H + \dots$$

$$\Rightarrow \mu = H + m * H$$

$$\cancel{\mu(t) = 1 - F(t+z) + \int_0^t} \qquad H(t) = 1 - F(t+z)$$

$$\mu(t) = H(t) + \int_0^t H(t-x)\, dm(x)$$

$$= \cancel{(1-F(t+z)) + \int}$$

By key Renewal thm

$$\lim_{t\to\infty} \mu(t) = \lim_{t\to\infty} H(t) + \frac{1}{\mu}\int_0^\infty H(x)\, dx$$

$$= \frac{1}{\mu}\int_0^\infty [1-F(x+z)]\, dx$$

$$= \frac{1}{\mu}\int_z^\infty [1-F(x)]\, dx$$

$$= 1 - \frac{1}{\mu}\int_0^z [1-F(x)]\, dx$$