4/11/2022

Martingales I

Refresher about conditional expectation

$\underline{X} = (X_1, \dots X_K)$ r. variables, Y r. var.

$$\mathbb{E}(Y|\underline{X}) = \mathbb{E}(Y|X_1 \dots X_K) \quad \text{is a rv!}$$

defined by 2 conditions

(1) $\mathbb{E}(Y|\underline{X}) = \hat{F}_Y(\underline{X})$ for some function $\hat{F}_Y$

(2) $\mathbb{E}[\mathbb{E}(Y|\underline{X})\, g(\underline{X})] = \mathbb{E}[Y g(\underline{X})]$

For any function g.

Explici

$$(\mathbb{E}(Y|\emptyset) = \mathbb{E}(Y))$$

$$\mathbb{E}(Y|c) = \mathbb{E}(Y) \text{ for a const } c$$

Explicit formula for discrete r.v.

$$p_{\underline{X},Y}(a,b) = \mathbb{P}(\underline{X}=a, Y=b)$$

$$\hat{f}_Y(a) = E(Y \mid \underline{X}=a)$$

$$= \frac{\sum_b b \, p_{\underline{X},Y}(a,b)}{\sum_{b'} p_{\underline{X},Y}(a,b')}$$

For r.v. with joint density

$$\hat{f}_Y(a) = \frac{\int b \, f_{\underline{X}Y}(a,b) \, db}{\int f_{X,Y}(a,b') \, db'}$$

Tower property

$$\mathbb{E}[\mathbb{E}(Y \mid X_1, X_2) \mid X_1] = \mathbb{E}(Y \mid X_1)$$

~~Def Let $(X_n)_{n\geq 0}$, $(Y_n)_{n\geq 0}$ be two sequences of~~

~~Reparametrization~~ / change of variables.

$\phi_k : \mathbb{R}^k \to \mathbb{R}^k$ invertible

$$\mathbb{E}(Y \mid X_1 \ldots X_k) = \mathbb{E}(Y \mid \phi_k(X_1, \ldots X_k))$$

$\square$

(Intuition : ϕ_k does not change the amount of information.)

Also $\forall$ $h : \mathbb{R}^k \to \mathbb{R}^m$

$$\mathbb{E}(Y \mid X_1 \ldots X_k, h(X_1 \ldots X_k)) = \mathbb{E}(Y \mid X_1 \ldots X_k)$$

(side rmk

For all of these reasons it is convenient to introduce σ-algebras & $\mathcal{F}_k = \sigma(X_1 \ldots X_k)$

$$\mathbb{E}(Y \mid \mathcal{F}_k) = \mathbb{E}(Y \mid X_1 \ldots X_k)$$
$$= \mathbb{E}(Y \mid \phi_k(X_1 \ldots X_k))$$
$$= \ldots$$

Def Let $(X_n)_{n\geq 1}$, $(Y_n)_{n\geq 1}$ be two sequences of r.v's. We say that $(Y_n)_{n\geq 0}$ is a MG wrt $(X_n)_{n\geq 0}$ if $\forall n$:

(1) $\mathbb{E}|Y_n| < \infty$

(2) $\mathbb{E}(Y_{n+1} | X_1 \dots X_n) = Y_n$.

In particular $Y_n = f_n(X_0 \dots X_n)$.

~~the~~

Example 1 Random walk (symmetric)

$(X_i)_{i\geq 1}$ iid $\mathbb{P}(X_i = +1) = \mathbb{P}(X_i = -1) = 0$

$$Y_n = \sum_{i=1}^{n} X_i = Y_{n-1} + X_n$$

$$\mathbb{E}(Y_{n+1} | X_1 \dots X_n) = \mathbb{E}(Y_n + X_{n+1} | X_1 \dots X_n)$$

generalizes to iid (X_i) $\mathbb{E}|X_i| < \infty$

$\mathbb{E}(X_i) = 0$.

$= Y_n + \mathbb{E}(X_{n+1} \mid X_1 \ldots X_n)$

$= Y_n$

<u>Example 2</u> $(X_i)_{i \geq 1}$ indep. $\mathbb{E}(e^{\theta X_1}) < \infty$ $\forall \theta \in$

$\overset{\text{def}}{=} e^{\phi(\theta)}$

$$Y_n = C_n \cdot \exp\left(\theta \sum_{i=1}^{n} X_i\right)$$

$$\mathbb{E}[Y_{n+1} \mid X_1 \ldots X_n] = C_{n+1}\, e^{\theta \sum_{i=1}^{n} X_i}\, \mathbb{E}[e^{\theta X_{n+1}} \mid X_1 \ldots X_n]$$

$$= \frac{C_{n+1}}{C_n} Y_n\, e^{\phi(\theta)}$$

a MG if $C_n = e^{-n\phi(\theta)}$

$$Y_n = e^{\theta \sum_{i=1}^{n} X_i - n\phi(\theta)}.$$

~~eg (X~~

example $X_i = \begin{cases} +1 & \text{w prob } p \\ -1 & \text{w prob } q = 1-p \end{cases}$

~~e $\phi(\theta)$ =~~

$$e^{\phi(\theta)} = pe^{\theta} + qe^{-\theta}$$

$$Y_n = (pe^{\theta} + qe^{-\theta})^{-n} e^{\theta \sum_{i=1}^{n} X_i}$$

$X_i \sim N(\mu, \sigma^2)$ $\phi(\theta) = \mu\theta + \frac{1}{2}\sigma^2\theta^2$

$$Y_n = \exp\left\{\theta \sum_{i=1}^{n} (X_i - \mu) - \frac{n\sigma^2\theta^2}{2}\right\}$$

Example $(X_n)_{n\geq 1}$ a $\overbrace{\text{Markov Chain}}^{\text{discr time}}$ on countable space S

$\mathbb{P}(X_{n+1} = j \mid X_n = i) = p_{ij}$ tr prob

$\mathbb{P}(X_{n+1} = j \mid X_1 \ldots X_n) = p_{X_n, j}$

Want to construct a MG $Y_n = \psi(X_n)$

$$\mathbb{E}(Y_{n+1} \mid X_1 \ldots X_n) = \mathbb{E}[\psi(X_{n+1}) \mid X_n] =$$

$$= \sum_{j \in S} \psi(j)\, p_{ij} \Big|_{X_n = i}$$

Harmonic functions

So I need $\psi(x) = \sum_{y \in S} p_{xy}\, \psi(y)$

Example X_n lazy random walk in $\mathbb{Z}^2 \cap [-L, L]^2$ ~~stob~~ stopped at the bound

$$p_{xy} = \begin{cases} \frac{1}{4}(1-\epsilon) & \text{if } \|x-y\|_2 = 1 \\ \epsilon & \text{if } x = y \end{cases}$$

for $|x|_\infty < L$

on the bd $p_{xx} = 1$

~~$\psi(x) \neq$~~

$$\psi(x) = \frac{1}{4}(1-\epsilon) \sum_{y : d(x,y) = 1} \psi(y) + \epsilon\, \psi(x)$$

$y \in \partial x$

$$\sum_{y \in \partial x} [\psi(y) - \psi(x)] = 0$$

$\Delta \psi(x) = 0$

Example 4 [Doob MG]

Z a rv $\mathbb{E}|Z|<\infty$

$(X_n)_{n\geq 1}$ r.v.

defin $Y_n = \mathbb{E}[Z|X_1..X_n]$

Indeed $X_1^n := (X_1..X_n)$

$$\mathbb{E}[Y_{n+1}|X_1^n] = \mathbb{E}[\mathbb{E}[Z|X_1^n X_{n+1}]|X_1^n]$$

$$= \mathbb{E}[Z|X_1^n] = Y_n.$$

Sub example

Color Chromatic

Define SUB-SUPER MG

sub MG $\mathbb{E}(Y_n^+)<\infty$

$$\mathbb{E}(Y_{n+1}|X_1..X_n) \geq Y_n.$$