4/13/2022

Martingale concentration inequality (Azuma-Hoeffding)

Thm Let $(Y_n)_{n\geq 0}$ be a MG wrt $(X_n)_{n\geq 1}$
($Y_0 = \mathbb{E}(Y_n)$ $\forall n$ is a const.) and assume
$\forall k \geq 1$ $\mathbb{P}(|Y_k - Y_{k-1}| \leq L_k) = 1$.

Then

$$\mathbb{P}(|Y_n - Y_0| \geq t) \leq 2\exp\left(-\frac{t^2}{2\sum_{k=1}^{n} L_k^2}\right).$$

Example 1 RW $S_n = \sum_{i=1}^{n} X_i$, ~~$Y_0 = 0$~~ S_0

(X_i) iid $|X_i| \leq L$. $\mathbb{E}X_i = \mu$

$Y_n := S_n - \mu n$

$|Y_n - Y_0| = |S_n - \mu n|$

$\mathbb{P}(|S_n - \mu n| \geq t) \leqslant 2e^{-\frac{t^2}{2L^2 n}}$

($\cancel{\text{cf CLT}}$ $\cancel{\lim_{n\to\infty}}$

equivalently

$\mathbb{P}\left(\frac{|S_n - \mu n|}{\sqrt{n}} \geqslant x\right) \leqslant 2e^{-\frac{x^2}{2L^2}}$

(compare with CLT $\cancel{\text{te}}$ $Var(X_1) = \sigma^2$)

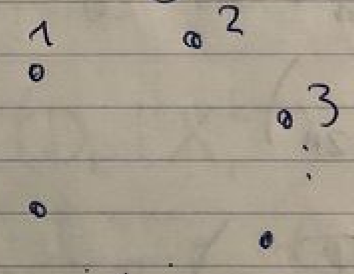
$\lim_{n\to\infty} \mathbb{P}\left(\frac{|S_n - \mu n|}{\sqrt{n}} \geqslant x\right) = 2\Phi\left(-\frac{x}{\sigma}\right)$

$\Phi(-t) = \int_{-\infty}^{-t} e^{-\frac{u^2}{2}} \frac{du}{\sqrt{2\pi}} \simeq \frac{e^{-t^2/2}}{t\sqrt{2\pi}}$

(Shamir Spencer 1987)

Example 2 Chromatic number of ~~ex~~ Erdös-Renyi Random graphs

$G \sim \mathcal{G}(n,p) \;=\; G = (V=[n], E)$



For each $i<j$

$\mathbb{P}((ij) \in E) = p$

and these events are indep

$N = \binom{n}{2}$ ~~Listed~~

List pairs $e_1 = (1,2),\ e_2 = (1,3) \ldots$

$\ldots\ e_N = (n-1, n)$

~~$X_k = \begin{cases} 1 & \text{if } e_k \in E \\ 0 & \text{o.w.} \end{cases}$~~

~~Z = chromatic number of G~~

Z = chromatic nb. of G.

How much does it fluctuate?

For all $i<j$

& $E_{ij} = \begin{cases} 1 & \text{if } (ij) \in E \\ 0 & \text{ow} \end{cases}$

$X_1 = (E_{12}, E_{13}, \dots \quad E_{1,n})$ ← edges adj

$X_2 = (0, E_{23}, \dots \quad E_{2n})$

$X_i = (0, \dots 0 E_{i,i+1}, E_{in})$

$X_{n-1} = (0, \quad 0, E_{n-1,n})$

$Z =$ chromatic nb

$Y_k = \mathbb{E}(Z \mid X_1 \dots X_k)$

$$|Y_k - Y_{k-1}| \leq \max_{v,v'} \Big| \mathbb{E}(Z \mid X_1 \dots X_k, X_{k+1} = v) - \mathbb{E}(Z \mid X_1 \dots X_k, X_{k+1} = v') \Big|$$

≤ 1

$$\mathbb{P}(|Z-\mathbb{E}(Z)| \geq t) \leq 2e^{-t^2/2n}$$

Proof of A-H

Define MG differences $D_k = Y_k - Y_{k-1}$

$$\mathbb{E}(D_k \mid X_1^{k-1}) = 0$$

$$F_k(\theta) = \mathbb{E}\left[e^{\theta(Y_k - Y_0)}\right] = \mathbb{E}\left[e^{\theta D_k + \theta(Y_{k-1} - Y_0)}\right]$$

$$= \mathbb{E}\left[\mathbb{E}\left[e^{\theta D_k + \theta(Y_{k-1} - Y_0)} \mid X_1^{k-1}\right]\right] =$$

$$= \mathbb{E}\left[e^{\theta(Y_{k-1} - Y_0)} \mathbb{E}\left[e^{\theta D_k} \mid X_1^{k-1}\right]\right]$$

Lemma

~~$\exp(x) \leq x$~~ $\mathbb{E}D = 0$

Consider rv $|D| \leq L$ const. Then $\mathbb{E}(e^{\theta D}) \leq e$

Proof of Lemma

$$\phi(\theta) = \log \mathbb{E} e^{\theta D}$$

$$\phi'(\theta) = \frac{\mathbb{E} D e^{\theta D}}{\mathbb{E} e^{\theta D}} \qquad \phi'(0) = 0$$

$$\phi''(\theta) = \frac{\mathbb{E} D^2 e^{\theta D}}{\mathbb{E} e^{\theta D}} - \left(\frac{\mathbb{E} D e^{\theta D}}{\mathbb{E} e^{\theta D}} \right)^2 \leq L^2$$

$$\Rightarrow \phi(\theta) \leq \frac{1}{2} L^2 \theta^2$$

QED

$$F_k(\theta) \leq \mathbb{E}\left[e^{\theta(Y_{k-1} - Y_0)} e^{L^2\theta^2/2} \right]$$

$$= e^{L^2\theta^2/2} F_{k-1}(\theta)$$

$$F_k(\theta) \leq e^{L^2\theta^2 n/2}$$

$$\cancel{\mathbb{P}(|Y_n - Y_0| \geq t) = 1}$$

$$\mathbb{P}(Y_n - Y_0 \geq t) \leq e^{t\theta - L^2\theta^2 n/2}$$

minimizing over θ . .