4/18/2022

Martingale convergence

$(Y_n)_{n\geq 1}$ a MG (wrt seq $(X_n)_{n\geq 1}$)

In general $\lim_{n\to\infty} Y_n$ does not exist

<u>example</u> (X_i) iid $\mathbb{P}(X_i=+1) = \mathbb{P}(X_i=-1)=\frac{1}{2}$

$$Y_n = \sum_{i=1}^{n} X_i \quad \text{does not converge!}$$

However if Y_n has some "boundedness" properties then it converges.

- Will state a general thm + special cases

Thm Let (Y_n) be a sub MG and assume $\exists M < \infty$ st $\mathbb{E}(Y_n^+) \leq M \ \forall n$.

Then ~~...~~

(1) $\exists Y_\infty$ such that $Y_n \xrightarrow{as} Y_\infty$ as $n \to \infty$.

(2) If $\mathbb{E}|Y_0| < \infty$ then $\mathbb{E}|Y_\infty| < \infty$

(3) If (Y_n) Uniformly integrable then $\mathbb{E}|Y_n - Y_\infty| \to 0$ $(Y_n \xrightarrow{L^1} Y_\infty)$.

[(Y_n) UI if

$$\lim_{B \to \infty} \sup_n \mathbb{E}\{|Y_n| \mathbb{1}_{|Y_n| \geq B}\} = 0$$

Sufficient condition

$$\sup_n \mathbb{E}(|Y_n|^\alpha) \leq C$$

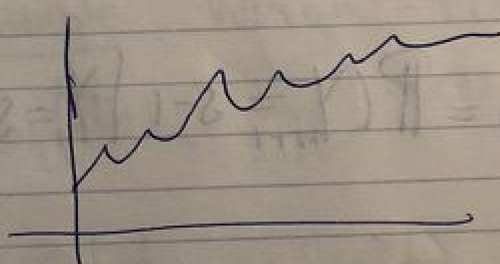
for some $\alpha > 1$, $C < \infty$.]

Intuition

- A sub MG is something that grows ave

$$\mathbb{E}(Y_{n+1} | X_1^n) \geq Y_n$$

- Condition $\mathbb{E}(Y_n^+) \leq M$ means th
cannot diverge to $+\infty$



Corollary Let Y_n be a MG with
$\sup_n \mathbb{E}|Y_n| \leq M$. Then $\exists Y_\infty$ such th

$$\cancel{\lim_n} \quad Y_n \xrightarrow{a.s.} Y_\infty$$

If $\sup_n \mathbb{E}|Y_n|^\alpha \leq M'$, we also have

$$Y_n \xrightarrow{L^1} Y_\infty$$

Will prove MG convergence theorem ~~For~~ for martingales $(Y_n)_{n \geq 1}$ under the stronger assumption

$$\sup_{n \geq 1} \mathbb{E}(Y_n^2) \leq M < \infty \quad \text{(bdd in } L^2\text{)}$$

~~For compactness~~

Note that $\forall k \geq 0,\ n \geq 1$

$$0 \leq \mathbb{E}[(Y_{n+k} - Y_n)^2] =$$

$$= \mathbb{E}(Y_{n+k}^2) - 2\mathbb{E}(Y_{n+k} Y_n) + \mathbb{E}(Y_n^2) =$$

$$= \mathbb{E}(Y_{n+k}^2) - \mathbb{E}(Y_n^2)$$

$\Rightarrow$ $\mathbb{E}(Y_n^2)$ is monotone non-decr bdd

$$\lim_{n \to \infty} \mathbb{E}(Y_n^2) = B \leq M$$

$$\lim_{n \to \infty} \sup_{k \geq 0} \mathbb{E}[(Y_{n+k} - Y_n)^2] = 0$$

Would like to use this to prove

$$Y_n(\omega) \to Y_\infty(\omega) \quad \text{for almost every } \omega.$$

<u>Thm</u> [Kolmogorov ineq]

Let $(Z_n)_{n\geq 1}$ be a subMG $Z_n \geq 0 \ \forall$

Then $\forall t > 0$

$$\mathbb{P}(\max(Z_1, \ldots, Z_n) \geq t) \leq \frac{\mathbb{E}(Z_n)}{t}.$$

Note: this is very interesting because it bounds the max by the last element!

Corollary

$$\mathbb{P}\left(\max_{0 \leq k \leq m} |Y_{n+k} - Y_n| \geq t\right) \leq \frac{\mathbb{E}[(Y_{n+m} - Y_n)^2]}{t^2}$$

Indeed $(Y_{n+k} - Y_n)^2 = Z_k$ is a non-neg sub

& getting back to MG convergence

$\epsilon_n \downarrow 0$

$$\limsup_{m \geq 0} \mathbb{P}\left(\max_{0 \leq k \leq m} |Y_{n+k} - Y_n| \geq t\right) \leq \frac{\epsilon_n}{t^2}$$

$$\mathbb{P}\left(\sup_{k, l \geq n} |Y_k - Y_l| \geq t\right) \leq \frac{4\epsilon_n}{t^2}$$

$$\mathbb{P}\left(\lim_{n \to \infty} \sup_{k, l \geq n} |Y_k - Y_l| \geq t\right) = 0$$

$\Rightarrow (Y_k)$ is a.s Cauchy

$\Rightarrow Y_k \xrightarrow{a.s.} Y_\infty$

$\square$

Corollary Any non-negative MG Y_n converges a.s. to a limit.

Example ~~S_n~~ ~~$Y_n = x_0 + \sum_{i=1}^{n} X_i$~~

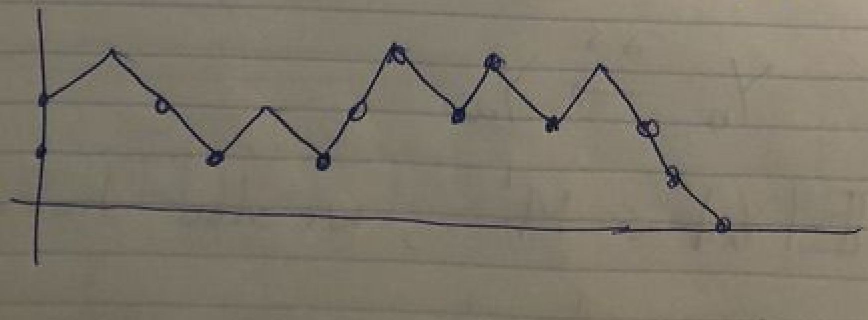
~~$x_0 \in \mathbb{N}$~~ Consider Y_n a rw on $\mathbb{Z}_{\geq 0}$ that stops at 0 $(X_n) = (Y_n)$

$$\mathbb{P}(Y_{n+1} = s+1 \mid Y_n = s) = \mathbb{P}(Y_{n+1} = s-1 \mid Y_n = s) = \frac{1}{2}$$

for $s \geq 1$

$$\mathbb{P}(Y_{n+1} = 0 \mid Y_n = 0) = 1$$

$Y_0 = x_0 > 0$ $x_0 \in \mathbb{Z}_{>0}$



$Y_n \xrightarrow{a.s} Y_\infty$

$$\mathbb{P}(Y_\infty = k) = \mathbb{P}(\lim_{n\to\infty} Y_n = k) = 0 \quad \forall k >$$

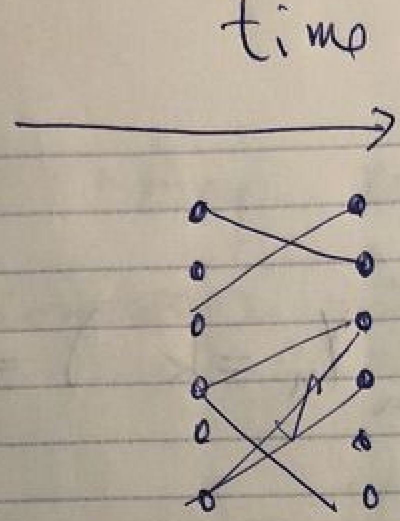
Hence $\mathbb{P}(Y_\infty = 0) = 0$

We proved $Y_n \xrightarrow{a.s} 0$ (sooner or later hits

$\boxed{\text{Note}}$ this applies to any Gambling schem

Example Wright-Fisher model of genetic drift

- Population with M individuals
- ~~Each po~~ Two alleles of a gene: A, a.
- Discrete time generations $n, n+1$ ..
- At each generation, each indiv. is equally likely to descend from any one in prev.



$Y_0 = k$

$Y_n \in \{0, \dots m\}$

Call Y_n = # individuals with allele A at gen n.

$$\mathbb{P}(Y_{n+1} = k \mid Y_1^n) = \text{Binom}\left(\frac{Y_n}{m}, m\right)$$

$$= \binom{m}{k}\left(\frac{Y_n}{m}\right)^k \left(1 - \frac{Y_n}{m}\right)^{m-k}$$

This is a MG

$$\mathbb{E}(Y_{n+1} \mid Y_1 \dots Y_n) = \mathbb{E}(Y_{n+1} \mid Y_n) = m \cdot \frac{Y_n}{m}$$

$$= Y_n.$$

$\phi \leq Y_n \leq m$. By DOM

$\lim_{n\to\infty} Y_n = Y_\infty$ a.s. $\mathbb{E}(|Y_n - Y_\infty|^q) \to 0 \quad \forall q$

$\mathbb{E}[(Y_{n+1} - Y_n)^2 | Y_1^n] = Var(Bin(\frac{Y_n}{m}, m)) =$

$= \cancel{m} \frac{Y_n}{\cancel{m}} (1 - \frac{Y_n}{m})$

$\mathbb{E}[(Y_{n+1} - Y_n)^2] = \frac{1}{m} \mathbb{E}[Y_n(m - Y_n)]$

$0 = \lim_{n\to\infty} \mathbb{E}[(Y_{n+1} - Y_n)^2] = \frac{1}{m} \mathbb{E}[Y_\infty(m - Y_\infty)]$

$\mathbb{P}(Y_\infty \in \{0, m\}) = 1$

$\mathbb{E} Y_\infty =$ $k = \mathbb{E} Y_0 = \mathbb{E} Y_\infty = m \mathbb{P}(Y_\infty = \cancel{km}$

$k =$ $\boxed{\mathbb{P}(Y_\infty = m) = \frac{k}{m}}$

The allele fixates ~~k~~ ~~[illegible]~~ to A with prob. $\frac{k}{m}$