[Ross makes the difference
stopping time $\mathbb{P}(T<\infty)=1$
random time ow

Stopping times

Def $(X_i)_{i\geq 1}$ a sequence of r.v's

A rv taking values in ~~{1,2,...}~~ $\{0,1,2,\ldots\}\cup\{\infty\}$
is a STOPPING TIME for (X_i) if
$\forall n \quad \exists F_n$

$$\{T\leq n\} = F_n(X_1^n)$$

At time n, we can decide whether the stopping event happened by looking at past and present.

Example ① $X_i \in \mathbb{R}$ $\quad B\subseteq\mathbb{R}$

$$T_B := \min\{n : X_n \in B\}$$

Will start using & σ-algebra notation

$\mathcal{F}_n = \sigma(X_1 \dots X_n)$

<u>Theorem</u> $(Y_n)_{n\geq 1}$ a (sub) MG wrt $\mathcal{F}_n$

T stopping time wrt $\mathcal{F}_n$.

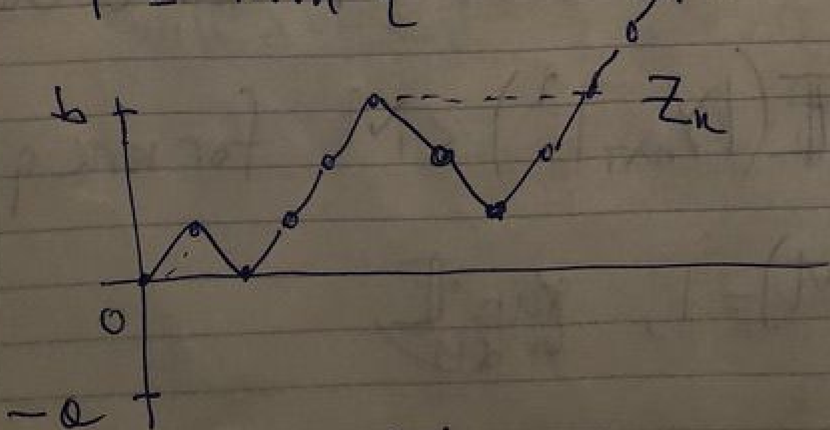
Define $Z_n = Y_{n \wedge T}$

Then (Z_n) is a (sub) MG.

<u>Example</u> $X_i = \begin{cases} +1 & \text{w prob } 1/2 \\ -1 & \text{w prob } 1/2 \end{cases}$

$Y_n = \sum_{i=1}^{n} X_i$ RW start at 0 $\qquad a, b \in \mathbb{Z}_{>0}$

$T = \min\{n : Y_n \leq -a \text{ or } Y_n \geq b\}$



$Z_n = Y_{n \wedge T}$

Z_n is bdd.

$Z_n \xrightarrow{a.s.} Z_\infty$ and in L^1

$\mathbb{P}(Z_\infty \in \{-a, b\}) = 1$

$0 = \mathbb{E} Z_0 = \mathbb{E} Z_\infty = -a\,\mathbb{P}(Z_\infty = -a) + b\,\mathbb{P}(Z_\infty = b)$

$$\begin{cases} \mathbb{P}(Z_\infty = -a) = \dfrac{b}{a+b} = \mathbb{P}(T_{-a} < T_b) \\ \mathbb{P}(Z_\infty = b) = \dfrac{a}{a+b} = \mathbb{P}(T_{-a} > T_b) \end{cases}$$

<u>Corollary</u> Assume $\mathbb{P}(T < \infty) = 1$ and either

(1) $\sup_n \mathbb{E}(|Y_{n \wedge T}|^q) < M$ for some $q > 1$

or (2) $\mathbb{P}(T \leq M) = 1$, $\cancel{\sup_{n \leq M} \mathbb{E}}$

Then $\mathbb{E} Y_T = \mathbb{E} Y_0$