Stat 300A Theory of Statistics

Homework 7: Solutions

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- Solutions should be complete and concisely written. Please, use a separate sheet (or set of sheets) for each problem.
- We will be using Gradescope (`https://www.gradescope.com`) for homework submission (you should have received an invitation) - no paper homework will be accepted. Handwritten solutions are still fine though, just make a good quality scan and upload it to Gradescope.
- You are welcome to discuss problems with your colleagues, but should write and submit your own solution.

P3.1 from Lehmann, Romano, *Testing Statistical Hypotheses.*

Let $X_1, \dots, X_n$ an i.i.d. sample from $\mathcal{N}(\xi, \sigma^2)$.

(i) If $\sigma = \sigma_0$ (known), there exists a UMP test for testing $H : \xi \leq \xi_0$ against $\xi > \xi_0$ which rejects when $\sum(X_i - \xi_0)$ is large.

(ii) If $\xi = \xi_0$ (known), there exists a UMP test for testing $H : \sigma \leq \sigma_0$ against $\sigma > \sigma_0$ which rejects when $\sum(X_i - \xi_o)^2$ is too large.

Solution:

(i) Let us first derive the optimal test for $H : \xi = \xi_0$ against $K : \xi = \xi_1$, where $\xi_1 > \xi_0$. The optimal test is of course the Neyman-Pearson (NP) test.

Writing $f_\xi(\boldsymbol{X}) = \prod_{i=1}^n f_\xi(X_i)$ for the likelihood under ξ, we know that the NP test rejects for large values of:

$$\begin{aligned}\frac{f_{\xi_1}}{f_{\xi_0}}(\boldsymbol{X}) &= \exp\left(-\frac{1}{2\sigma_0^2}\sum_{i=1}^n \left[(X_i - \xi_1)^2 - (X_i - \xi_0)^2\right]\right) \\ &= \exp\left(\frac{2(\xi_1 - \xi_0)\sum_{i=1}^n X_i - n(\xi_1^2 - \xi_0^2)}{2\sigma_0^2}\right)\end{aligned}$$

Observe that since $\xi_1 > \xi_0$, rejecting for large values of $\frac{f_{\xi_1}}{f_{\xi_0}}(\boldsymbol{X})$ must be equivalent to rejecting for large values of $\sum_{i=1}^n X_i$, hence also for large values of $T(\boldsymbol{X}) = \sum_{i=1}^n (X_i - \xi_0)$, i.e. the NP test rejects when $T(\boldsymbol{X}) \geq c$ for some constant c (note that all distributions here have density w.r.t. Lebesgue measure hence we do not need to be careful about $>$ or $\geq$).

The constant c is of course determined so as to control the type-I error, i.e. with Φ the standard Normal pdf we require:

$$\alpha = \mathbb{P}_{\xi_0}\left[T(\boldsymbol{X}) \geq c\right] = 1 - \Phi\left(\frac{c}{\sqrt{n}}\right)$$

In other words $c = \sqrt{n} z_{1-\alpha}$, with $z_{1-\alpha}$ the $1 - \alpha$ quantile of the standard Normal distribution.

To summarize: The NP test for $H : \xi = \xi_0$ versus $K : \xi = \xi_1$ rejects when:

$$T(\boldsymbol{X}) \geq \sqrt{n} z_{1-\alpha}$$

We now make two observations: First, this is still a valid test for $H : \xi \leq \xi_0$, since for any $\xi \leq \xi_0$, we have that:

$$\mathbb{P}_\xi \left[T(\boldsymbol{X}) \geq c\right] = 1 - \Phi\left(\frac{c - n(\xi - \xi_0)}{\sqrt{n}}\right) \leq 1 - \Phi\left(\frac{c}{\sqrt{n}}\right) = \alpha$$

Thus it must also be UMP for $H : \xi \leq \xi_0$ vs. $K : \xi = \xi_1$.

Second its form does not depend on the specific choice of $\xi_1 > \xi_0$, hence it must be UMP against any $\xi_1 > \xi_0$, i.e. it is UMP for $H : \xi \leq \xi_0$ against $K : \xi > \xi_0$.

Note that this argument could have been shortened by sufficiency and monotone likelihood ratio considerations.

(ii) We proceed as in part (i). Fix $\sigma_1 > \sigma_0$ and let us derive the NP test for $H : \sigma = \sigma_0$ vs. $K : \sigma = \sigma_1$. The likelihood ratio is:

$$\frac{f_{\sigma_1}}{f_{\sigma_0}}(\boldsymbol{X}) = \frac{\sigma_0^n}{\sigma_1^n} \exp\left(\left(\frac{1}{2\sigma_0^2} - \frac{1}{2\sigma_1^2}\right) \sum_{i=1}^n (X_i - \xi_0)^2\right)$$

Since $\left(\frac{1}{2\sigma_0^2} - \frac{1}{2\sigma_1^2}\right)$, we see that the NP test rejects for large values of $T(\boldsymbol{X}) = \sum_{i=1}^n (X_i - \xi_0)^2$.

Since under $H : \sigma = \sigma_0$ it holds that $\frac{T(\boldsymbol{X})}{\sigma_0^2} \sim \chi_n^2$, we see that the NP test takes rejects the null when $T(\boldsymbol{X}) \geq c$, where $c = \sigma_0^2 \chi_{n,1-\alpha}^2$ with $\chi_{n,1-\alpha}^2$ the $1 - \alpha$ quantile of the Chi-squared distribution with n degrees of freedom.

Now note that for $\sigma \leq \sigma_0$:

$$\mathbb{P}_\sigma \left[T(\boldsymbol{X}) \geq c\right] = \mathbb{P}_\sigma \left[\frac{T(\boldsymbol{X})}{\sigma_0^2} \geq \chi_{n,1-\alpha}^2\right] \leq \mathbb{P}_\sigma \left[\frac{T(\boldsymbol{X})}{\sigma^2} \geq \chi_{n,1-\alpha}^2\right] = \alpha$$

So the proposed test is even most powerful for $H : \sigma \leq \sigma_0$ against $K : \sigma = \sigma_1$. Its form does not depend on the specific choice of $\sigma_1 > \sigma_0$, hence it is UMP for $H : \sigma \leq \sigma_0$ versus $K : \sigma > \sigma_0$.

P3.2 from Lehmann, Romano, *Testing Statistical Hypotheses.*

Let $X = (X_1, \dots, X_n)$ an i.i.d. sample from the uniform distribution $U[0, \theta]$.

(i) For testing $H : \theta \leq \theta_0$ against $K : \theta > \theta_0$ any test is UMP at level α for which $\mathsf{E}_{\theta_0}\phi(X) = \alpha$, $\mathsf{E}_{\theta}\phi(X) \leq \alpha$ for $\theta \leq \theta_0$ and $\phi(x) = 1$ when $\max\{x_1, \dots, x_n\} > \theta_0$.

(ii) For testing $H : \theta = \theta_0$ against $K : \theta \neq \theta_0$, a unique UMP test exists and is given by $\phi(x) = 1$ when $\max\{x_1, \dots, x_n\} > \theta_0$ or $\max\{x_1, \dots, x_n\} \leq \theta_0 \alpha^{1/n}$ and $\phi(x) = 0$ otherwise.

Solution:

(i) Let us fix $\theta_1 > \theta_0$ and show that the specified test is a Neyman-Pearson test for $H : \theta = \theta_0$ versus $K : \theta = \theta_1$. For convenience we also write $T(\boldsymbol{X}) = \max\{X_1, \dots, X_n\}$ and $\frac{f_{\theta_1}}{f_{\theta_0}}(\boldsymbol{X})$ for the likelihood ratio:

$$\frac{f_{\theta_1}}{f_{\theta_0}}(\boldsymbol{X}) = \frac{\theta_0^n}{\theta_1^n} \delta_{\theta_0, \theta_1}(T(\boldsymbol{X}))$$

We defined:

$$\delta_{\theta_0,\theta_1}(u) = \begin{cases} \infty & \text{, for } u \in (\theta_0, \theta_1] \\ 1 & \text{, for } u \in [0, \theta_0] \end{cases}$$

This defines $\delta_{\theta_0,\theta_1}(T(X))$ on sets with both P_{θ_0} and P_{θ_1} probability equal to 1.

Any test ϕ of the form specified specified in the question, is a test with size α, i.e. $\mathsf{E}_{\theta_0}\phi(X) = \alpha$ and furthermore it rejects when $\frac{f_{\theta_1}}{f_{\theta_0}}(\boldsymbol{X}) > \frac{\theta_0^n}{\theta_1^n}$ and accepts when $\frac{f_{\theta_1}}{f_{\theta_0}}(\boldsymbol{X}) < \frac{\theta_0^n}{\theta_1^n}$ (this last statement is vacuous since P_{θ_0} and P_{θ_1} almost surely this never happens). By the sufficient condition of the NP theorem (Theorem 3.2.1. in TSH), it must be most powerful and since $\theta_1 > \theta_0$ was arbitrary, it is UMP for $H : \theta = \theta_0$ against $K : \theta > \theta_0$.

Finally this test is also valid for $H : \theta \leq \theta_0$, since by assumptions it also satisfies $\mathsf{E}_{\theta}\phi(X) \leq \alpha$ for $\theta \leq \theta_0$. Thus it is UMP for $H : \theta \leq \theta_0$ against $K : \theta > \theta_0$.

(ii) We start by showing that the specified test is indeed UMP.

Let us first check that under θ_0:

$$\mathbb{P}_{\theta_0}\left[T(\boldsymbol{X}) \leq \theta_0 \alpha^{1/n}\right] = \prod_{i=1}^n \mathbb{P}_{\theta_0}\left[X_i \leq \theta_0 \alpha^{1/n}\right] = \left(\alpha^{1/n}\right)^n = \alpha$$

Hence by our argument in part (i), the specified test is a UMP test for testing $H : \theta = \theta_0$ against $K : \theta > \theta_0$.

Let us now check that it is UMP at the other tail too. Thus let us test θ_0 vs θ_1 with $0 < \theta_1 < \theta_0$ and show it is most powerful.

Here the Likelihood ratio takes the form:

$$\frac{f_{\theta_1}}{f_{\theta_0}}(\boldsymbol{X}) = \frac{\theta_0^n}{\theta_1^n} \tilde{\delta}_{\theta_1, \theta_0}(T(\boldsymbol{X}))$$

Where we define $\tilde{\delta}_{\theta_1,\theta_0}$ on the $P_{\theta_0}, P_{\theta_1}$-support of $T(\boldsymbol{X})$ as:

$$\tilde{\delta}_{\theta_1,\theta_0}(u) = \begin{cases} 0 & \text{, for } u \in (\theta_1, \theta_0] \\ 1 & \text{, for } u \in [0, \theta_1] \end{cases}$$

We see that the test given in the question rejects when $\frac{f_{\theta_1}}{f_{\theta_0}}(\boldsymbol{X}) > \frac{\theta_0^n}{\theta_1^n}\frac{1}{2}$ and accepts when $\frac{f_{\theta_1}}{f_{\theta_0}}(\boldsymbol{X}) < \frac{\theta_0^n}{\theta_1^n}\frac{1}{2}$. It furthermore has size α, thus by the sufficient condition of the NP theorem (Theorem 3.2.1. in TSH), it must be most powerful.

It remains to check *uniqueness*. Let us first find the unique most powerful test of θ_0 against $\theta_1 = \alpha^{1/n}\theta_0$. The necessary condition of Theorem 3.2.1. now implies that there exists a k such that the most powerful test rejects for $\tilde{\delta}_{\theta_1,\theta_0}(T(\boldsymbol{X})) > k$ and accepts for $\tilde{\delta}_{\theta_1,\theta_0}(T(\boldsymbol{X})) < k$. This k clearly cannot be < 0, for otherwise we would not have size α. It also clearly cannot be > 1, for otherwise the test would have power 0.

Let us consider the following cases for k. When we write almost surely below we require the statement to hold P_{θ_0} and P_{θ_1} almost surely.

- If $k \in (0,1)$, then the test takes exactly the form of accepting when $\tilde{\delta}_{\theta_1,\theta_0}(T(\boldsymbol{X})) = 0$ and rejecting when it is $= 1$. But this is equivalent to rejecting when $T(\boldsymbol{X}) \leq \theta_1 = \alpha^{1/n}\theta_0$ and accepting otherwise.
- If $k = 0$, then the test still rejects at $T(\boldsymbol{X}) \leq \theta_1 = \alpha^{1/n}\theta_0$ and it remains to consider what happens when $\tilde{\delta}_{\theta_1,\theta_0}(T(\boldsymbol{X})) = 0$. However almost surely the test needs to accept on this event, since otherwise the test will have size $> \alpha$.
- If $k = 1$, the test accepts when $\tilde{\delta}_{\theta_1,\theta_0}(T(\boldsymbol{X})) = 0$ and we need to check what happens when it is $= 1$. Clearly power is (strictly) maximized by always rejecting when $k = 1$ (vs. potentially not rejecting with some positive probability), yet still has the correct size α.

To recap: The necessary condition of the NP theorem for θ_0 vs $\theta_0\alpha^{1/n}$ forces the test to take the form (at least almost everywhere w.r.t. Lebesgue measure) of rejecting when $T(\boldsymbol{X}) \leq \theta_0\alpha^{1/n}$ and accepting when $T(\boldsymbol{X}) \in (\theta_0\alpha^{1/n}, \theta_0]$. The same condition hence must also apply to the UMP test of $H : \theta = \theta_0$ vs $K : \theta \neq \theta_0$.

We still need to check what the test does for $T(\boldsymbol{X}) > \theta_0$. It is clear that the test should always reject then, since this happens with probability 0 under P_{θ_0}, hence this does not influence the size of the test, yet increases power under any $\theta_1 > \theta_0$. To be more precise: Fixing any $\theta_1 > \theta_0$ we see that power under this θ_1 will be strictly maximized if we always reject for $T(\boldsymbol{X}) \in (\theta_0, \theta_1]$ and now note that this holds for any $\theta_1 > \theta_0$,

P3.9 from Lehmann, Romano, *Testing Statistical Hypotheses.*

Let X distributed according to $P_\theta, \theta \in \Omega$ and let T sufficient for θ. If $\phi(X)$ is any test of a hypothesis concerning θ, then $\psi(T)$ given by $\psi(t) = \mathsf{E}[\phi(X) \mid T = t]$ is a test depending on T only and its power is identical with that of $\phi(X)$.

Solution:

First let us observe that since $\phi(\cdot)$ is a (potentially randomized) test, it means that $\phi(\cdot) \in [0, 1]$. Hence also $\psi(\cdot) = \mathsf{E}[\phi(X) \mid T = \cdot] \in [0, 1]$ (to be more precise: there exists a version of the conditional expectation with this property). Furthermore $\psi(t)$ is a well-defined statistic, since T is sufficient for θ and hence the conditional expectation $\mathsf{E}[\phi(X) \mid T = t] = \mathsf{E}_\theta[\phi(X) \mid T = t]$ does not depend on θ.

Now take any $\theta \in \Omega$ and note that by the tower property (iterated expectation) we get:

$$\mathsf{E}_\theta\left[\psi(T)\right] = \mathsf{E}_\theta\left[\mathsf{E}_\theta\left[\phi(X) \mid T\right]\right] = \mathsf{E}_\theta[\phi(X)]$$

But the right-hand side is just the power function of the test ϕ. So in particular, if $\phi(X)$ is a level α test, then so is $\psi(T)$, i.e. if we let $H \subset \Omega$ denote the (parameters corresponding to the) null hypothesis being tested, then:

$$\sup_{\theta \in H} \mathsf{E}_\theta\left[\psi(T)\right] = \sup_{\theta \in H} \mathsf{E}_\theta\left[\phi(X)\right] \leq \alpha$$

If we also let $K \subset \Omega$ for the alternative, then for any $\theta \in K$ the two tests have the same power.