Stat 300A Theory of Statistics

Homework 8 Solutions

Due on December 5, 2018

- Solutions should be complete and concisely written. Please, use a separate sheet (or set of sheets) for each problem.
- We will be using Gradescope (`https://www.gradescope.com`) for homework submission (you should have received an invitation) - no paper homework will be accepted. Handwritten solutions are still fine though, just make a good quality scan and upload it to Gradescope.
- You are welcome to discuss problems with your colleagues, but should write and submit your own solution.

Problems on hypothesis testing

Solve problems 3.32, 4.2 and 4.19 from Lehmann, Romano, *Testing Statistical Hypotheses.*

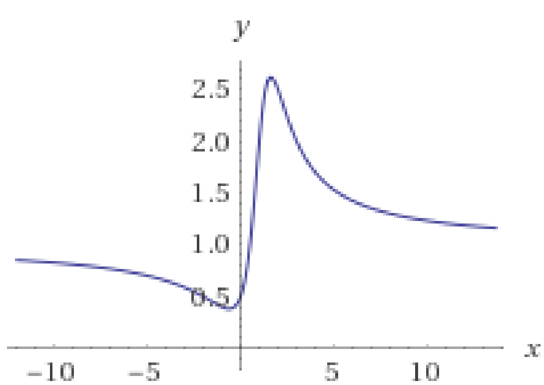
TSH 3.32 solution (sketch)

The likelihood ratio is:

$$r(x) = \frac{p_{\theta_1}(x)}{p_{\theta_0}(x)} = \frac{1 + (x - \theta_0)^2}{1 + (x - \theta_1)^2}$$

by Neyman-Pearson, the MP test rejects when $r(x) > k$ with k chosen such that $P_{\theta_0}(r(x) > k) = \alpha$. Since X has a continuous distribution, the behavior at k doesn't matter.

We plot an example with $\theta_0 = 0$ and $\theta_1 = 1$.



Notice that $r(x) \to 1$ as $x \to \pm\infty$, unlike for normal distributions and other MLR examples that we have seen. It is clear that as we vary θ_1, the set $r(x) \geq \alpha$ changes, so there is no UMP test for $\theta = 0$ versus $\theta > 0$.

To characterize the set of distributions, we set $r(x) = k$ for various values of θ_0, θ_1 and k and look at the shape. We find that we can get any set of the form:

- $[a, b]$ for $-\infty \leq a < b \leq \infty$
- $[-\infty, a] \cap [b, \infty]$ for $-\infty < a < b < \infty$

TSH 4.2 solution

$$\begin{aligned} P_\theta(\hat{\alpha} \le \alpha) &= P_\theta(x \in S_{\alpha'} \text{ for some } \alpha' \le \alpha) \\ &= P_\theta(x \in S_\alpha) \;\text{ by the nested sets assumption} \\ &\ge P_{\theta_0}(x \in X_\alpha) \;\text{ by unbiasedness} \\ &= \alpha \end{aligned}$$

TSH 4.19 solution (sketch)

Let $T' = \sum_{i=1}^N T(X_i)$ and notice that (N, T') are the sufficient statistics of a multi-parameter exponential family:

$$P(N,T) \propto e^{N \log(\lambda/C(\theta)) + Q(\theta)T'} h(X,N).$$

We now wish to apply theorem 4.4.1 from TSH, but the theorem doesn't apply because we don't know that the parameter space $\{(\log(\lambda/C(\theta)), Q(\theta)) : \lambda > 0, \theta \in \Theta\}$ is convex. Looking at the proof of theorem 4.4.1, we find that we only need $\omega_j = \{(\log(\lambda/C(\theta)), Q(\theta)) : \lambda > 0, \theta = Q^{-1}(c)\}$ to contain a rectangle, which it does because λ can be any positive number. Thus the proof of theorem 4.4.1 goes through in our situation, as desired.