Stat 310A/Math 230A Theory of Probability

Final

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Solutions should be complete and concisely written. Please, mark clearly the beginning and end of each problem.

You have 3 hours but you are not required to solve all the problems!

Just solve those that you can solve within the time limit. Points assigned to each problem are indicated in parenthesis. I recommend to look at all problems before starting.

For any clarification on the text, one of the TAs will be outside the room.

You can consult textbooks (Dembo, Billinglsely, Williams, Durrett) and your notes. You cannot use computers, and in particular you cannot use the web. You can cite theorems (propositions, corollaries, lemmas, etc.) from Amir Dembo's lecture notes by number, and exercises you have done as homework by number as well. Any other non-elementary statement must be proved!

Problem 1

(20 points)

Let X, Y be two random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and assume they are independent and identically distributed.

(*a*) Prove that, for any $\varepsilon > 0$, there exists $x_0 \in \mathbb{R}$ such that

$$\mathbb{P}\big(X \in (x_0 - \varepsilon, x_0 + \varepsilon)\big) > 0\,. \tag{1}$$

(*b*) Prove that, for any $\varepsilon > 0$,

$$\mathbb{P}\big(|X - Y| < \varepsilon\big) > 0\,. \tag{2}$$

Problem 2

(20 points)

Let $\{X_n\}_{n\ge 1}$ be a sequence of independent and identically distributed random variables, with $\mathbb{P}(X_n \ge 0) = 1$.

(*a*) Prove that, if $\mathbb{E}(X_1) < \infty$, then, almost surely.

$$\limsup_{n\to\infty} \frac{X_n}{n} = 0\,. \tag{3}$$

[Hint: Use Borel-Cantelli, and the fact that the $\{X_n\}_{n\ge 1}$ are identically distributed to estimate the resulting sum.]

(*b*) Prove that, if $\mathbb{E}(X_1) = \infty$, then, almost surely,

$$\limsup_{n\to\infty} \frac{X_n}{n} = \infty\,. \tag{4}$$

Problem 3 (20 points)

Let $(\nu_n)_{n\in\mathbb{N}}$ and ν_∞ be probability measures on $((0,1],\mathcal{B}_{(0,1]})$. Denote by $F_n(t)=\nu_n((0,t])$ and $F_\infty(t)=\nu_\infty((0,t])$ the corresponding distribution functions. Prove that $\nu_n \overset{w}{\Rightarrow} \nu_\infty$ (ν_n converges weakly to ν_∞) if and only if

$$\lim_{n\to\infty}\int_{(0,1]}\left|F_n(t)-F_\infty(t)\right|\mathrm{d}t=0\,. \tag{5}$$

Problem 4 (20 points)

(*a*) Let $\{Z_n\}_{n\in\mathbb{N}}$, Z_∞ be a sequence of random variables on the probability space $(\Omega,\mathcal{F},\mathbb{P})$. Assume that $Z_n \overset{p}{\to} Z_\infty$. Let $n_1(\ell)$, $n_2(\ell)$ be two sequences indexed by $\ell\in\mathbb{N}$, with $n_1(\ell),n_2(\ell)\to\infty$ as $\ell\to\infty$.

Prove that $Z_{n_1(\ell)}-Z_{n_2(\ell)} \overset{p}{\to} 0$ as $\ell\to\infty$.

(*b*) Let $\{X_i\}_{i\geq 1}$ be independent and identically distributed random variables with $\mathbb{E}(X_1)=0$, $\mathbb{E}(X_1^2)=1$, and define the partial sums $S_n=\sum_{i\leq n}X_i/\sqrt{n}$. Remember that the central limit theorem implies $S_n \overset{d}{\Rightarrow} G\sim\mathsf{N}(0,1)$.

Is it possible to construct a random variable S_∞ such that $S_n \overset{p}{\to} S_\infty$? Prove your answer.

Problem 5 (20 points)

Let $\{\nu_n\}_{n\in\mathbb{N}}$, ν_∞ be probability measures on $(\mathbb{R}^m,\mathcal{B}_{\mathbb{R}^m})$. Also, define the class of rectangles by

$$\mathsf{R}_m\equiv\left\{A=[a_1,b_1]\times\cdots\times[a_m,b_m]:\ \ a_i<b_i\forall i\leq m\right\}. \tag{6}$$

(*a*) Recall that a Borel set A is a ν_∞-continuity set if $\nu_\infty(\partial A)=0$. Prove that $\lim_{n\to\infty}\nu_n(A)=\nu_\infty(A)$ for all rectangles $A\in\mathsf{R}_m$ that are ν_∞-continuity sets, if and only if $\nu_n \overset{w}{\Rightarrow} \nu_\infty$.

(*b*) Let $m=m_1+m_2$, for some integers $m_1,m_2\geq 1$, and assume that $\nu_n=\nu_n^{(1)}\times\nu_n^{(2)}$, where $\nu_n^{(a)}$ is a probability measure on $\mathbb{R}^{m_a}$. (In other words, if $\boldsymbol{Z}^{(n)}=(X_1^{(n)},\ldots,X_m^{(n)})\sim\nu_n$ is a random vector, then $(X_1^{(n)},\ldots,X_{m_1}^{(n)})\sim\nu_n^{(1)}$ is independent of $(X_{m_1+1}^{(n)},\ldots,X_m^{(n)})\sim\nu_n^{(2)}$.)

Assume that $\nu_n \overset{w}{\Rightarrow} \nu_\infty$. Prove that $\nu_\infty=\nu_\infty^{(1)}\times\nu_\infty^{(2)}$ (here $\nu_\infty^{(a)}$ is a probability measure on $\mathbb{R}^{m_a}$), and that $\nu_n^{(a)} \overset{w}{\Rightarrow} \nu_\infty^{(a)}$.