Stats 310A Midterm Solutions

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Problem 1

(a) For any $\epsilon > 0$, we have that

$$1 = \mathbf{P}(X \in \mathbb{R}) \leq \sum_{k \in \mathbb{Z}} \mathbf{P}(X \in (k\epsilon - \epsilon, k\epsilon + \epsilon)). \tag{1}$$

In particular, at least one of the terms of the sum must be non-zero, so the result follows with the corresponding $x_0 = k\epsilon$.

(b) Let $\epsilon > 0$ and x_0 such that $\mathbf{P}(X \in (x_0 - \epsilon/2, x + \epsilon/2)) > 0$. We then have that, since X, Y are iid,

$$\begin{aligned} \mathbf{P}(|X - Y| < \epsilon) &\geq \mathbf{P}(|X - x_0| < \epsilon/2, |Y - x_0| < \epsilon/2) && (2) \\ &= \mathbf{P}(X \in (x_0 - \epsilon/2, x_0 + \epsilon/2))^2 && (3) \\ &> 0 && (4) \end{aligned}$$

Problem 2

(a) Notice that for any $\epsilon > 0$, since $X_n \geq 0$ almost surely, we have that

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbf{P}\Big(\frac{X_n}{n} > \epsilon\Big) &= \sum_{n=1}^{\infty} \mathbf{P}\Big(\frac{X_1}{\epsilon} > n\Big) && (5) \\ &\leq \int_0^{\infty} \mathbf{P}\Big(\frac{X_1}{\epsilon} > t\Big)\,\mathrm{d}t && (6) \\ &= \mathbb{E}(X_1)/\epsilon && (7) \\ &< \infty. && (8) \end{aligned}$$

Thus, by Borel-Cantelli, we have that

$$\mathbf{P}\Big(\frac{X_n}{n} > \epsilon \text{ i. o.}\Big) = 0. \tag{9}$$

Taking a union of probability-0 events, we thus conclude

$$\mathbf{P}\Big(\limsup_{n\to\infty} \frac{X_n}{n} > 0\Big) = \mathbf{P}\Big(\bigcup_{r=1}^{\infty} \Big\{\frac{X_n}{n} > \frac{1}{r} \text{ i. o.}\Big\}\Big) = 0. \tag{10}$$

(b) If $\mathbb{E}(X_n) = \infty$, we have instead for every $K > 0$

$$\sum_{n=1}^{\infty} \mathbf{P}\Big(\frac{X_n}{n} > K\Big) = \sum_{n=1}^{\infty} \mathbf{P}\Big(\frac{X_1}{K} > n\Big) \tag{11}$$

$$\geq \int_1^{\infty} \mathbf{P}\Big(\frac{X_1}{K} > t\Big)\,\mathrm{d}t \tag{12}$$

$$\geq \mathbb{E}(X_1)/K - 1 \tag{13}$$

$$= \infty. \tag{14}$$

Since the X_n are mutually independent, the second Borel-Cantelli lemma yields that

$$\mathbf{P}\Big(\frac{X_n}{n} > K \text{ i. o.}\Big) = 1. \tag{15}$$

Taking an intersection of probability-1 events, we now have

$$\mathbf{P}\Big(\limsup_{n\to\infty} \frac{X_n}{n} = \infty\Big) = \mathbf{P}\Big(\bigcap_{K=1}^{\infty} \Big\{\frac{X_n}{n} > K \text{ i. o.}\Big\}\Big) = 1. \tag{16}$$

Problem 3 Suppose that $\nu_n \Rightarrow \nu_\infty$, and so let C be the set of continuity points of F_∞. The complement of this set is at most countable, and so in particular has Lebesgue measure 0.

Now, $|F_n(t) - F_\infty(t)| \leq 2$ and $F_n(t) \to F_\infty(t)$ on C, so the dominated convergence theorem yields that

$$\int_{(0,1]} |F_n(t) - F_\infty(t)|\,\mathrm{d}t = \int_C |F_n(t) - F_\infty(t)|\,\mathrm{d}t \to 0 \tag{17}$$

To see the converse, suppose that $\int_{(0,1]} |F_n(t) - F_\infty(t)|\,\mathrm{d}t \to 0$. This means that $F_n(t) \to F_\infty(t)$ on a set A with measure 1.

Suppose that $F_n(t_0)$ doesn't converge to $F_\infty(t_0)$ for t_0 a continuity point of F_∞. This means that there exists an $\epsilon > 0$ such that, either $F_n(t_0) > F_\infty(t_0) + \epsilon$ infinitely often or $F_n(t_0) < F_\infty(t_0) - \epsilon$ infinitely often (or both).

We consider this first case, so assume that $F_\infty(t_0) < F_n(t_0) - \epsilon$ infinitely often. Since t_0 is a continuity point of F_∞, let $\delta > 0$ be such that, for any $t \in [t_0, t_0 + \delta]$, $F_\infty(t) < F_\infty(t_0) + \epsilon/2$.

Since F_n is non-decreasing, we have for any $t \in [t_0, t_0 + \delta]$ that

$$F_\infty(t) < F_\infty(t_0) + \frac{\epsilon}{2} \tag{18}$$

$$< F_n(t_0) - \frac{\epsilon}{2} \tag{19}$$

$$\leq F_n(t) - \frac{\epsilon}{2}. \tag{20}$$

But this means that F_n fails to converge to F_∞ on the positive-measure set $[t_0, t_0 + \delta]$, which is a contradiction.

The second case is similar, assuming that $F_\infty(t_0) > F_n(t_0) + \epsilon$ infinitely often. Since t_0 is a continuity point of F_∞, let $\delta > 0$ be such that, for any $t \in [t_0 - \delta, t_0]$, $F_\infty(t) > F_\infty(t_0) - \epsilon/2$

Since F_n is non-decreasing, we have for any $t \in [t_0 - \delta, t_0]$ that

$$F_\infty(t) > F_\infty(t_0) - \frac{\epsilon}{2} \tag{21}$$

$$> F_n(t_0) + \frac{\epsilon}{2} \tag{22}$$

$$\geq F_n(t) + \frac{\epsilon}{2}. \tag{23}$$

But this means that F_n fails to converge to F_∞ on the positive-measure set $[t_0-\delta, t_0]$, which is also a contradiction.

Hence, it must be the case that $F_n(t) \to F_\infty(t)$ for any t which is a continuity point of F_∞. Hence, $\nu_n \Rightarrow \nu_\infty$

Note: A cleaner but less elementary argument is the following:

Since both integrals denote the same area, we have that

$$\int_0^1 |F_n(t) - F_\infty(t)| \,\mathrm{d}t = \int_0^1 |F_n^{-1}(s) - F_\infty^{-1}(s)| \,\mathrm{d}s \tag{24}$$

$$= \mathbb{E}|X_n - X_\infty|, \tag{25}$$

where $X_n \sim F_n$ are the random variables constructed in the proof of the Skorokhod representation theorem.

In particular, since L_1 convergence implies distributional convergence, if $\mathbb{E}|X_n - X_\infty| \to 0$, then $\nu_n \Rightarrow \nu_\infty$.

Conversely, by the Skorokhod construction, if $\nu_n \Rightarrow \nu_\infty$, then $X_n \xrightarrow{\text{a.s.}} X_\infty$, where both random variables are bounded by 1. The dominated convergence theorem thus yields that $\mathbb{E}|X_n - X_\infty| \to 0$.

Problem 4

(a) We have that, for any $\epsilon > 0$,

$$\mathbf{P}(|Z_{n_1(\ell)} - Z_{n_1(\ell)}| > \epsilon) \le \mathbf{P}(|Z_{n_1(\ell)} - Z_\infty| > \epsilon/2) + \mathbf{P}(|Z_{n_2(\ell)} - Z_\infty| > \epsilon/2) \to 0. \tag{26}$$

(b) Notice that

$$S_{2n} - S_n = \frac{1}{\sqrt{2n}} \sum_{i=1}^{2n} X_i - \frac{1}{\sqrt{n}} \sum_{j=1}^{n} X_j \tag{27}$$

$$= \frac{1-\sqrt{2}}{\sqrt{2}} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i + \frac{1}{\sqrt{2}} \frac{1}{\sqrt{n}} \sum_{j=n+1}^{2n} X_j \tag{28}$$

Since the X_i are independent, the central limit theorem yields the joint convergence in distribution of $(\sum_{i=1}^n X_i/\sqrt{n}, \sum_{j=n+1}^{2n} X_j/\sqrt{n})$ to (G, G'), a pair of independent standard normals. In particular,

$$S_{2n} - S_n \xrightarrow{d} \frac{1-\sqrt{2}}{\sqrt{2}} G + \frac{1}{\sqrt{2}} G' \sim \mathcal{N}(0, 2-\sqrt{2}). \tag{29}$$

In particular, $S_{2n} - S_n$ does not converge in probability to 0, and so by part (a), S_n cannot converge in probability.

Problem 5

(a) The Portmanteau theorem yields that if $\nu_n \Rightarrow \nu_\infty$, then $\nu_n(A) \to \nu_\infty(A)$ for *all* ν_∞-continuity sets, and so in particular for all ν_∞-continuity rectangles.

Conversely, suppose $\nu_n(A) \to \nu_\infty(A)$ for all rectangles $A \in R_m$ which are ν_∞-continuity sets. To show $\nu_n \Rightarrow \nu_\infty$, we need to show that for all bounded continuous functions $f : \mathbb{R}^m \to$

$\mathbb{R}$, we have $\nu_n(f) \to \nu_\infty(f)$. It suffices to assume f is non-negative, and $\|f\|_\infty \leq 1$. To start, observe that for any simple function of the form $h = \sum_{i=1}^k b_i \mathbb{I}_{A_i}$, with $A_i \in R_m$ for all $1 \leq i \leq k$, and $0 \leq b_i < \infty$ for all $1 \leq i \leq k$, we have $\nu_n(h) \to \nu_\infty(h)$. So to finish, it suffices to show that for all $\varepsilon > 0$, there exists simple functions $\ell_\varepsilon, u_\varepsilon$ of the previously described form, such that $\ell_\varepsilon \leq f \leq u_\varepsilon$, and $\nu_\infty(u_\varepsilon) - \nu_\infty(\ell_\varepsilon) \leq \varepsilon$.

Towards this end, fix $\varepsilon > 0$. There exists some large, finite rectangle $A_0 \in R_m$ such that $\nu_\infty(A_0^c) \leq \varepsilon/2$. As A_0 is compact, f is uniformly continuous on A_0. Thus there exists $\delta > 0$ such that for all $x, y \in A_0$, $\|x - y\|_\infty \leq \delta$ implies $|f(x) - f(y)| \leq \varepsilon/2$. As the set of atoms of ν_∞ is at most countable, there exists some collection of rectangles $A_1, \ldots, A_k \in R_m$, such that $A_0 = A_1 \cup \cdots \cup A_k$, the interiors of the rectangles $A_1^\circ, \ldots, A_k^\circ$ are disjoint, and for all $1 \leq i \leq k$, A_i has all side lengths at most δ. Given this collection, define $b_i^\ell := \inf_{x \in A_i} f(x)$, $b_i^u := \sup_{x \in A_i} f(x)$. Define

$$\ell_\varepsilon := \sum_{i=1}^k b_i^\ell \mathbb{I}_{A_i^\circ},$$

$$u_\varepsilon := \mathbb{I}_{A_0^c} + \sum_{i=1}^k b_i^u \mathbb{I}_{A_i}.$$

Using the facts that the interiors $A_1^\circ, \ldots, A_k^\circ$ are disjoint, $A_0 = A_1 \cup \cdots \cup A_k$, f is non-negative, and $\|f\|_\infty \leq 1$, we have $\ell_\varepsilon \leq f \leq u_\varepsilon$. Moreover, recalling the definition of δ, and using the fact that A_i has all side lengths at most δ for all $1 \leq i \leq k$, we have $u_i^\ell - b_i^\ell \leq \varepsilon/2$ for all $1 \leq i \leq k$. To finish, observe that since $A_i \in R_m$, we have $\nu_\infty(A_i) = \nu_\infty(A_i^\circ)$, so that

$$\begin{aligned}
\nu_\infty(u_\varepsilon) - \nu_\infty(\ell_\varepsilon) &\leq \nu_\infty(A_0^c) + \sum_{i=1}^k (b_i^u - b_i^\ell) \nu_\infty(A_i^\circ) \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \sum_{i=1}^k \nu_\infty(A_i^\circ) \\
&\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon,
\end{aligned}$$

as desired.

(b) By definition, for any bounded continuous function $f : \mathbb{R}^m \to \mathbb{R}$, we have

$$\mathbb{E} f(\mathbf{Z}^{(n)}) \to \mathbb{E} f(\mathbf{Z}^{(\infty)}).$$

In particular, given a bounded continuous function $g : \mathbb{R}^{m_1} \to \mathbb{R}$, we may define $f : \mathbb{R}^m \to \mathbb{R}$ by $f(x_1, \ldots, x_m) := g(x_1, \ldots, x_{m_1})$. As f is bounded and continuous, we obtain

$$\mathbb{E} g(X_1^{(n)}, \ldots, X_{m_1}^{(n)}) \to \mathbb{E} g(X_1^{(\infty)}, \ldots, X_{m_1}^{(\infty)}).$$

This shows that $\nu_n^{(1)} \overset{w}{\Rightarrow} \nu_\infty^{(1)}$. The proof for $\nu_n^{(2)}$ is the same. To show that ν_∞ is a product measure, take bounded continuous functions $g_1 : \mathbb{R}^{m_1} \to \mathbb{R}$, $g_2 : \mathbb{R}^{m_2} \to \mathbb{R}$. Define $f : \mathbb{R}^m \to \mathbb{R}$ by $f(x_1, \ldots, x_m) := g_1(x_1, \ldots, x_{m_1}) g_2(x_{m_1+1}, \ldots, x_m)$. Note f is also bounded continuous, and thus (using also the assumption that the ν_n are product measures)

$$\mathbb{E} g_1(X_1^{(n)}, \ldots, X_{m_1}^{(n)}) \mathbb{E} g_2(X_{m_1+1}^{(n)}, \ldots, X_m^{(n)}) \to \mathbb{E} g_1(X_1^{(\infty)}, \ldots, X_{m_1}^{(\infty)}) g_2(X_{m_1+1}^{(\infty)}, \ldots, X_m^{(\infty)}).$$

But by weak convergence of the individual $\nu_n^{(a)}$, we also have that the limit is equal to

$$\mathbb{E}g_1(X_1^{(\infty)}, \dots, X_{m_1}^{(\infty)})\mathbb{E}g_2(X_{m_1+1}^{(\infty)}, \dots, X_m^{(\infty)}).$$