Stat 310A/Math 230A Theory of Probability

Homework 2 Solutions

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Exercises on measurable functions and Lebesgue integration

Exercise [1.2.14]

The same method works for all four parts.

1. Since $\mathcal{B} = \sigma(\{(-\infty, \alpha] : \alpha \in \mathbb{R}\})$, it follows from Theorem 1.2.11 that X is measurable with respect to the right hand side (RHS), which hence also contains the left hand side (LHS). But the RHS is generated by elements of the σ-algebra on the LHS, so the LHS contains the RHS as well.

2. For $1 \le i \le n$, each X_i is by Theorem 1.2.11 measurable with respect to the RHS. Therefore, the RHS contains the LHS. Again, the RHS is generated by sets from the LHS, so the latter contains the former.

3. Exactly the same method applies.

4. Since each X_k is measurable with respect to the RHS, the latter contains the LHS. By definition $\sigma(X_k, k \le n)$ is contained in the LHS for each n, hence so is the union of these collections, implying that the LHS contains the RHS as well.

Exercise [1.2.20]

1. If g is l.s.c. and x_n is a sequence of points that converge to x and such that $g(x_n) \le a$ for all n then necessarily also $g(x) \le \liminf_n g(x_n) \le a$. So, we see that $\{x : g(x) \le a\}$ is closed.

2. From part (a) we see that $g^{-1}((-\infty, a])$ is a closed set for l.s.c. g and $g^{-1}((-\infty, a))$ is an open set for u.s.c. g, hence both are in $\mathcal{B}_{\mathbb{R}^n}$. Since $(-\infty, a]$, $a \in \mathbb{R}$ generate $\mathcal{B}$ (see Exercise 1.1.17), as do $(-\infty, a)$, $a \in \mathbb{R}$, it follows (from Theorem 1.2.11) that any such g is a Borel function.

3. Since continuous functions are also l.s.c. we use part (b) here.

Exercise [1.2.30]

Since

$$\{\omega : \limsup_{n\to\infty} X_n(\omega) \le X_\infty(\omega)\} = \{\omega : \limsup_{n\to\infty} (X_n(\omega) - X_\infty(\omega)) \le 0\}\,,$$

it suffices to consider the case of $X_\infty = 0$. By the definition of $\limsup$,

$$\{\omega : \limsup_{n\to\infty} X_n(\omega) \le 0\} = \bigcap_{\epsilon>0} \{\omega : \limsup_{n\to\infty} X_n(\omega) < \epsilon\} = \bigcap_{\epsilon>0} \bigcup_{n=1}^{\infty} C_n\,,$$

for $C_n = \cap_{k\ge n}\{\omega : X_k(\omega) < \epsilon\}$. Since the sets C_n are non-decreasing in n it follows from our assumption that for any $\epsilon > 0$,

$$1 = \mathbf{P}(\bigcup_{n=1}^{\infty} C_n) = \lim_{m\to\infty} \mathbf{P}(\bigcup_{n=1}^{m} C_n) = \lim_{m\to\infty} \mathbf{P}(C_m)\,.$$

Hence, there exists N such that $\mathbf{P}(C_n) > 1 - \epsilon$ for all $n \ge N$ and the set C_N^c is the required event A from the statement of this exercise.

Exercise [1.2.31]

Expanding on Example 1.2.4 we know that each simple function is a measurable function from $(\Omega, \mathcal{F})$ to $(\mathbb{R}, \mathcal{B})$, hence belongs to the (larger) collection $m\mathcal{F}$ of all $\overline{\mathbb{R}}$-valued measurable functions on $(\Omega, \mathcal{F})$. Further, by Theorem 1.2.23 the latter collection is closed under pointwise limits. We complete the proof upon observing that $m\mathcal{F}$ is the minimal collection with these two properties since any $f \in m\mathcal{F}$ is the pointwise limit of some sequence f_n of simple fucntions (for example, take $f_n = -n \vee ((2^{-n}[2^n f]) \wedge n)$).

Exercise [1.2.49]

1. If x is a point of discontinuity of F, then $F(x) - F(x^-) > 0$, so by monotonicity, for each $\epsilon > 0$ also $F(x+\epsilon) - F(x-\epsilon) > 0$. On the other hand, if x is outside the support of F then for all $\epsilon > 0$ sufficiently small, $F(x+\epsilon) = F(x-\epsilon)$, by the definition of the support and the monotonicity of F. Now let (a, b) be an open interval in the complement of the support. Then, for each $x \in (a, b)$, there is an open interval (a_x, b_x) such that $x \in (a_x, b_x) \subset (a, b)$ and F is constant on (a_x, b_x). If further x is an isolated point of the support of F then there is an interval (x, b_x) outside the support of F, and F is constant over this interval. Also, there is an interval (a_x, x) outside the support, and F is constant over this interval as well. Since x is in the support, the two constants must be different, which implies that $F(x) > F(x^-)$, or in other words, that F has a discontinuity at x.

2. Let $\mu = \mathcal{P}_X$, a probability measure on $(\mathbb{R}, \mathcal{B})$ with the corresponding distribution function $F_X = F$. In the definition of the support of μ, the word "neighborhood" can be replaced without loss by "interval", since every open neighborhood of x contains an interval which contains x. Now if x is in the support of μ, then
$$F(x+\epsilon) - F(x-\epsilon) \;\geq\; \mu(x-\epsilon, x+\epsilon) > 0$$
so x is in the support of F and if x is in the support of F, then $\mu(x - 2\epsilon, x + 2\epsilon) > 0$, so x is in the support of μ.

Optional exercises

Exercise [1.2.15]

1. Let $\mathcal{C} = \{B \in \mathcal{F} : \inf_{A\in\mathcal{A}} \mathbf{P}(A\Delta B) = 0\}$. If $B \in \mathcal{A}$, taking $A = B$ we see that $B \in \mathcal{C}$ as well. So, it suffices to show that $\mathcal{C}$ is a σ-algebra (for then $\mathcal{C}$ contains $\mathcal{F} = \sigma(\mathcal{A})$). Since $A\Delta B = A^c \Delta B^c$ and $\mathcal{A}$ is closed with respect to taking complements, the same applies for $\mathcal{C}$. It remains to show that if $B_n \in \mathcal{C}$ then $B = \cup_{i=1}^{\infty} B_i$ must also be in $\mathcal{C}$. To this end, fix $\epsilon > 0$ and choose m so that $\mathbf{P}(B \setminus \bigcup_{i=1}^{m} B_i) < \epsilon/2$. Next, choose $\{A_i\}_{i\leq m} \subseteq \mathcal{A}$ so that $\mathbf{P}(A_i \Delta B_i) < \epsilon/(2m)$. Since
$$(\bigcup_{i\leq m} A_i)\Delta(\bigcup_{i\leq m} B_i) \subseteq \bigcup_{i\leq m} (A_i \Delta B_i)\,,$$
taking $A = \cup_{i\leq m} A_i$ which is in $\mathcal{A}$, we have that for $\hat{B} = \cup_{i\leq m} B_i$,
$$\mathbf{P}(A\Delta B) \leq \mathbf{P}(B \setminus \hat{B}) + \mathbf{P}(A\Delta\hat{B}) < \epsilon\,.$$
Since $\epsilon > 0$ is arbitrary, it follows that $B \in \mathcal{C}$ and we are done.

2. It suffices to show the property for X non-negative which we do in two steps. First, choose positive integer n large enough so that $X(\omega) \leq n$ almost surely and $2^{-n} < \epsilon$. This implies that $|X - f_n(X)| \leq \epsilon$ a.s. for $f_n(\cdot)$ as in the proof of Proposition 1.2.6. Next let $B_k = \{\omega : k2^{-n} < X(\omega) \leq (k+1)2^{-n}\}$ for

$k = 0, \ldots, n2^n - 1$, which are all in $\sigma(\mathcal{A})$. By part (a) we have for any $\delta > 0$ some sets $A_k \in \mathcal{A}$ such that $\mathbf{P}(A_k \Delta B_k) < \delta$. Taking $\delta = \epsilon/(n2^n)$ the random variable $Y = \sum_{k=0}^{n2^n-1} k2^{-n} I_{A_k}$ is such that

$$\mathbf{P}(|X - Y| > \epsilon) \le \mathbf{P}(f_n(X) \ne Y) \le \sum_{k=0}^{n2^n-1} \mathbf{P}(A_k \Delta B_k) < \epsilon \,,$$

with the simple function Y having the stated property.

Exercise [1.2.28]

1. As $g_*(x,\delta) = -[(-g)^*(x,\delta)]$ it suffices to show that $g^*(x,\delta)$ is l.s.c. To this end, fix $\delta > 0$, and let $B(x,\delta) = \{y : |y - x| < \delta\}$. Using the notation $(v)^\epsilon = \min(v - \epsilon, 1/\epsilon)$ and fixing $\epsilon > 0$, by definition $g(z) \ge (g^*(x))^\epsilon$ for some $z \in B(x,\delta)$. With $\eta = \delta - |z - x| > 0$ note that $z \in B(y,\delta)$ whenever $y \in B(x,\eta)$, so for any such y we have by the definition of $g^*(y)$ that $g^*(y) \ge g(z) \ge (g^*(x))^\epsilon$. Consequently, $\liminf_{y \to x} g^*(y) \ge (g^*(x))^\epsilon$. Finally, since $(v)^\epsilon \uparrow v$ as $\epsilon \downarrow 0$, for any $v \in [-\infty, \infty]$, we arrive at the stated conclusion that $\liminf_{y \to x} g^*(y) \ge g^*(x)$.

2. Let $\mathbf{D} = \mathbf{D}_- \cup \mathbf{D}_+$ denote the set of points at which g is discountinuous, where $\mathbf{D}_- = \{x : \exists \epsilon > 0$ and $x_n \to x$ such that $g(x_n) \le g(x) - \epsilon\}$, $\mathbf{D}_+ = \{x : \exists \epsilon > 0$ and $x_n \to x$ such that $g(x_n) \ge g(x) + \epsilon\}$. If $x \in \mathbf{D}_-$ then for any k,
$$g_*(x, k^{-1}) \le \sup_n g(x_n) < g(x) \le g^*(x, k^{-1}) \,.$$
Considering $k \uparrow \infty$ this inequality results with $\mathbf{D}_- \subseteq \mathbf{D}_g$. Similarly, if $x \in \mathbf{D}_+$ then for any k,
$$g_*(x, k^{-1}) \le g(x) < \inf_n g(x_n) \le g^*(x, k^{-1}) \,,$$
implying in the limit $k \uparrow \infty$ that $\mathbf{D}_+ \subseteq \mathbf{D}_g$. Conversely, if $x \in \mathbf{D}_g$ then there exists c such that
$$\sup_k g_*(x, k^{-1}) < c < \inf_k g^*(x, k^{-1}) \,.$$
Hence, $\liminf_n g(x_n) < c < \limsup_n g(x_n)$ for some $x_n \to x$. That is, x is a point of discontinuity of g, i.e. $x \in \mathbf{D}$.

3. The functions $g^*(x, k^{-1})$ and $g_*(x, k^{-1})$ are by part (a) semi-continuous, hence Borel measurable (see Exercise 1.2.18). It thus follows from Theorem 1.2.23 that so are $h^-(x) = \sup_k g_*(x, k^{-1})$ and $h^+(x) = \inf_k g^*(x, k^{-1})$. By definition $A_{r,q} = \{x : h^-(x) \le r, q \le h^+(x)\}$ is a Borel set, hence so is the countable union $\mathbf{D}_g$ of $A_{r,q}$ over $r < q$, $r, q \in \mathcal{Q}$.