Stat 310A/Math 230A Theory of Probability

Homework 8 Solutions

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Exercises on characteristic functions

Exercise [3.3.10]

1. Denoting by $\Phi_X(\theta)$ the ch.f. of X, since X and $\widetilde{X}$ are i.i.d., the ch.f. of $-\widetilde{X}$ is $\Phi_X(-\theta) = \overline{\Phi_X(\theta)}$. Hence, by Lemma 3.3.8,
$$\Phi_Z(\theta) = \Phi_X(\theta)\overline{\Phi_X(\theta)} = |\Phi_X(\theta)|^2 \geq 0\,.$$

2. If $U = X - \widetilde{X}$ for some i.i.d. X and $\widetilde{X}$, then by part (a), its ch.f. $\Phi_U(\theta)$ must be a real-valued non-negative function. Recall that the ch.f. of the uniform random variable on (a, b) is $\Phi_U(\theta) = e^{i\theta(a+b)/2}\sin(c\theta)/(c\theta)$ for $c = (b-a)/2$ (see Example 3.3.7). This function is real-valued only when $a = -b$ and even then $\sin(b\theta) = -1$ for $\theta = 3\pi/(2b) > 0$, leading to the stated conclusion.

Exercise [3.3.21]

1. Recall Example 3.3.7 showing that the Uniform Distribution on $(-1, 1)$, which is of bounded probability density function, has the ch.f. $\sin(\theta)/\theta$. Clearly, $\int_{\mathbb{R}}(|\sin\theta|/|\theta|)d\theta = \infty$ (consider $\theta \in [\pi n + \pi/4, \pi n + 3\pi/4]$, $n = 0, 1, 2, \ldots$ for which $|\sin\theta| \geq 1/\sqrt{2}$).

2. Recall Example 3.3.13 showing that the Cauchy distribution has the ch.f. $\exp(-|\theta|)$ which is not differentiable at $\theta = 0$.

Exercise [3.3.22]

Combining Lemma 3.3.8 and Example 3.3.7, we deduce that
$$\Phi_{S_n}(\theta) = (\sin\theta/\theta)^n\,.$$
For any $n \geq 2$, the integral $\int |(\sin\theta)/\theta|^n d\theta$ is finite. Thus, by the inversion formula (3.3.7), the r.v. S_n has the bounded continuous probability density function
$$\begin{aligned} f_{S_n}(s) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\theta s}(\sin\theta/\theta)^n d\theta \\ &= \frac{1}{\pi}\int_{0}^{\infty} \cos(\theta s)(\sin\theta/\theta)^n d\theta \end{aligned}$$
with the latter identity due to the fact that $(\sin\theta)/\theta$ is invariant under the change of variable $\theta \mapsto -\theta$. Since $S_n \leq n$, the continuous p.d.f. $f_{S_n}(\cdot)$ must be identically zero for $s > n$, yielding the stated conclusion that $\int_0^\infty \cos(\theta s)(\sin\theta/\theta)^n d\theta = 0$ for all $s > n \geq 2$.

Exercise [3.3.23]

$$\Phi_{n^{-1}\sum_{k=1}^{n} X_k}(\theta) = \prod_{k=1}^{n}\Phi_{X_k}(\theta/n) = \prod_{k=1}^{n} e^{-|\theta|/n} = e^{-|\theta|} = \Phi_{X_1}(\theta).$$
$$\therefore X_1 \stackrel{d}{=} \frac{1}{n}\sum_{k=1}^{n} X_k.$$

An exercise on weak convergence of measures

1. Indeed $|A_n| = \binom{n}{n/2}$ is finite and

$$\nu_n = \frac{1}{|A_n|} \sum_{\xi \in A_n} \delta_\xi \tag{1}$$

(this identity can be checked on the π-system $\mathcal{P} = \{N_\ell(\omega) : \ell \in \mathbb{N}, \omega \in \Omega\}$). Each δ_ξ is a probability measure, hence ν_n is a probability measure.

2. We claim that ν_n converges weakly to the uniform measure ν_∞, defined by

$$\nu_\infty(N_\ell(\omega)) = \frac{1}{2^\ell}\,. \tag{2}$$

In order to prove this fact, we will show that, for any bounded continuous function $h : \{0,1\}^{\mathbb{N}} \to \mathbb{R}$, $\lim_{n\to\infty} \nu_n(h) = \nu_\infty(h)$. Let us start by a function h measurable on $\sigma(\{N_\ell(\omega) : \omega \in \Omega\})$, i.e. depending only on the first ℓ coordinates of ω. For sugn a function we have $h(\omega) = h_\ell(\omega_1^\ell)$, $h_\ell : \{0,1\}^\ell \to \mathbb{R}$. Therefore

$$\nu_n(h) = \sum_{\xi_1^\ell} h_\ell(\xi_1^\ell)\nu_n(\{\omega : \omega_1^\ell = \xi_1^\ell\})\,. \tag{3}$$

But, letting $k = \xi_1 + \cdots + \xi_\ell$, we have

$$\lim_{n\to\infty} \nu_n(\{\omega : \omega_1^\ell = \xi_1^\ell\}) = \lim_{n\to\infty} \binom{n}{n/2}^{-1} \binom{n-\ell}{n/2-k} = \frac{1}{2^\ell}\,, \tag{4}$$

where the last equality is a straightforward application of Stirling formula. This proves the claim for $h \in \mathrm{m}\sigma(\{N_\ell(\omega) : \omega \in \Omega\})$.

Consider now a general bounded continuous function h, and let for $\ell \in \mathbb{N}$, $\hat{h}_\ell \in \mathrm{m}\sigma(\{N_\ell(\omega) : \omega \in \Omega\})$ be defined by $\hat{h}_\ell(\omega) = h(\omega_1^\ell, 0, 0, 0 \dots)$. By Fact 1, we have, for any probability measure μ, $|\mu(h) - \mu(\hat{h}_\ell)| \le \int |h(\omega) - \hat{h}_\ell(\omega)| \mathrm{d}\mu(\omega) \le \delta(\ell)$. Therefore

$$\begin{aligned} |\nu_n(h) - \nu_\infty(h)| &\le |\nu_n(h) - \nu_n(\hat{h}_\ell)| + |\nu_n(\hat{h}_\ell) - \nu_\infty(\hat{h}_\ell)| + |\nu_\infty(h) - \nu_\infty(\hat{h}_\ell)| && (5)\\ &\le |\nu_n(\hat{h}_\ell) - \nu_\infty(\hat{h}_\ell)| + 2\delta(\ell)\,. && (6) \end{aligned}$$

By letting $n \to \infty$, and using the above result, we get

$$\lim_{n\to\infty} |\nu_n(h) - \nu_\infty(h)| \le 2\delta(\ell)\,. \tag{7}$$

The thesis follows because ℓ can be taken arbitrarily large.