Stat 310A/Math 230A Theory of Probability

Midterm

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Solutions should be complete and concisely written. Please, use a separate booklet for each problem.

You have 3 hours but you are not required to solve all the problems!!!

Just solve those that you can solve within the time limit.

For any clarification on the text, one of the TA's will be outside the room, and Andrea in Packard 272.

You can consult textbooks and your notes. You cannot use computers, and in particular you cannot use the web. You can cite theorems (propositions, corollaries, lemmas, etc.) from Amir Dembo's lecture notes by number, and exercises you have done as homework by number as well. Any other non-elementary statement must be proved!

Problem 1 [5 points]

Prove that if a function $f : \mathbb{R} \to \mathbb{R}$ is right-continuous, then it is Borel (i.e. it is measurable with respect to the Borel σ-algebra $\mathcal{B}_{\mathbb{R}}$).
[Hint: Construct a sequence of measurable functions that converge pointwise to f.]

Problem 2 [10 points]

Let $\Omega = C([0,1])$ be the space of continuous functions over the interval $[0,1]$. Given two such functions ω_1, ω_2, we let their distance $d(\omega_1, \omega_2) = \sup_{t\in[0,1]} |\omega_1(t) - \omega_2(t)|$, and denote by $\mathcal{B}$ the Borel σ-algebra induced by the resulting topology. Also, for any $t \in [0,1]$, let $X_t : \Omega \to \mathbb{R}$ be defined by $X_t(\omega) = \omega(t)$. Let $\mathcal{F} = \sigma(\{X_t : t \in [0,1]\})$.

(*a*) Prove $\mathcal{F} \subseteq \mathcal{B}$.

(*b*) Prove $\mathcal{B} \subseteq \mathcal{F}$.

Problem 3 [20 points]

Let $(X_n)_{n\in\mathbb{N}}$, and X_∞ be real-valued random variables, on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(N_k)_{k\in\mathbb{N}}$ be a sequence of random variables taking values in $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, on the same space.

(*a*) Let $Z_k(\omega) = X_{N_k(\omega)}(\omega)$. Prove that Z_k is a random variable.

(*b*) Assume $X_n \xrightarrow{a.s.} X_\infty$ as $n \to \infty$, and $N_k \xrightarrow{a.s.} \infty$ as $k \to \infty$. Does this imply that the sequence $(Z_k)_{k\in\mathbb{N}}$ converges almost surely to X_∞? Prove your answer.

(*c*) Assume $X_n \xrightarrow{p} X_\infty$ as $n \to \infty$, and $N_k \xrightarrow{p} \infty$ as $k \to \infty$. Does this imply that the sequence $(Z_k)_{k\in\mathbb{N}}$ convergesin probability to X_∞? Prove your answer.

(*d*) Assume $X_n \xrightarrow{L_1} X_\infty$ as $n \to \infty$, and $N_k \xrightarrow{a.s.} \infty$ as $k \to \infty$. Does this imply that the sequence $(Z_k)_{k\in\mathbb{N}}$ converges in L_1 to X_∞? Prove your answer.

Problem 4 [10 points]

Given a Borel function $f : \mathbb{R}^n \to \mathbb{R}$, we say $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ if, defining $f_C(x) = f(x)\mathbb{I}_C(x)$, we have $f_C \in L^1(\mathbb{R}^n)$ for any compact set C. We also say $\varphi \in C_c^\infty(\mathbb{R}^n)$ if φ is infinitely many times differentiable, and has compact support.

(a) Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and assume that, for any $a_1 < b_1$, ..., $a_n < b_n$

$$\int f(x)\mathbb{I}_R(x)\, \mathrm{d}x = 0\,, \tag{1}$$

where $R = (a_1, b_1) \times \cdots \times (a_n, b_n)$. Prove that $f(x) = 0$ almost everywhere.

(b) Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and assume that, for any $\varphi \in C_c^\infty(\mathbb{R}^n)$,

$$\int f(x)\varphi(x)\, \mathrm{d}x = 0\,. \tag{2}$$

Prove that $f(x) = 0$ almost everywhere.

[Hint: It might be useful to know the following fact of analysis. For any $\varepsilon \in (0, 1)$, there exists a function $\eta_\varepsilon : \mathbb{R} \to [0, 1]$ such that $\eta_\varepsilon \in C^\infty(\mathbb{R})$, $\eta_\varepsilon(t) = 0$ for $|t| \geq 1$ and $\eta_\varepsilon(t) = 1$ for $|t| \leq 1 - \varepsilon$.]

Problem 5 [20 points]

For a function $f : \mathbb{R} \to \mathbb{R}$, and any $b \in \mathbb{R}$, we will denote by $\mathsf{T}_b f$ the translated of f, namely $\mathsf{T}_b f(x) = f(x-b)$. We also let $\mathrm{d}x$ denote the Lebesgue measure on $\mathbb{R}$, and $L^p(\mathbb{R}) = L^p(\mathbb{R}, \mathrm{d}x)$ the space of p-integrable Borel functions, $p \geq 1$ and by $\|f\|_p = [\int |f(x)|\, \mathrm{d}x]^{1/p}$ the corresponding (semi-)norm.

(a) Prove that, if $f \in L^p(\mathbb{R}, \mathrm{d}x)$, $p \in (1, \infty)$, then $\mathsf{T}_\varepsilon f \xrightarrow{L^p} f$ as $\varepsilon \to 0$.

(b) Assume $p, q \in (1, \infty)$ to be conjugate, i.e. $p^{-1} + q^{-1} = 1$, and let $f \in L^p(\mathbb{R})$ $g \in L^q(\mathbb{R})$. Prove that the convolution $f \star g$

$$f \star g(x) \equiv \int f(x - y)g(y)\, \mathrm{d}y\,, \tag{3}$$

is defined for every x, with $\sup_x |f \star g(x)| \leq \|f\|_p \|g\|_q < \infty$.

(c) Within the setting of the previous point, prove that $f \star g$ is uniformly continuous.

(d) Within the setting of the last two points, prove that $\lim_{x\to\infty} f \star g(x) = \lim_{x\to-\infty} f \star g(x) = 0$.