Stats 310A Midterm Solutions

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Problem 1 Let $f\colon \mathbb{R} \to \mathbb{R}$ be right-continuous. Define a sequence of functions f_n by

$$f_n(x) = f\Big(\frac{\lceil nx \rceil}{n}\Big) = \sum_{k \in \mathbb{Z}} f\Big(\frac{k}{n}\Big) \mathbb{I}_{((k-1)/n, k/n]}(x). \tag{1}$$

Each f_n is a countable sum of indicators and so is measurable. Moreover, for any x, we have that $\lceil nx \rceil / n \downarrow x$. By right-continuity of f, this implies that $f_n(x) \to f(x)$ and so that f is measurable.

Problem 2

(a) Notice that the evaluation operators $\mathrm{ev}_t(\omega) = \omega(t)$ satisfy, for any $\omega_1, \omega_2 \in \Omega$,

$$|\mathrm{ev}_t(\omega_1) - \mathrm{ev}_t(\omega_2)| = |\omega_1(t) - \omega_2(t)| \le d(\omega_1, \omega_2). \tag{2}$$

In particular, each ω_t is Lipschitz and so continuous with respect to the sup norm. We therefore have that each

$$\{\omega : X_t(\omega) < b\} = \{\omega : \omega(t) < b\} = \mathrm{ev}_t^{-1}((-\infty, b)) \tag{3}$$

is an open set and so in $\mathcal{B}$. Since such sets generate $\mathcal{F}$, we conclude that $\mathcal{F} \subseteq \mathcal{B}$.

(b) We first show that $C([0,1])$ with the sup norm has a countable dense subset. To construct this set, define

$$\mathcal{C}_n = \Big\{\phi \in C([0,1]) : \phi \text{ is a linear interpolation between } \{(k/n, q) : k = 0, \dots n; q \in \mathbb{Q}\}\Big\}. \tag{4}$$

Each such set is countable, so their union $\mathcal{D} = \bigcup_{n=1}^{\infty} \mathcal{C}_n$ is also countable.

To see that $\mathcal{D}$ is dense, let $f \in C([0,1])$ and let $\epsilon > 0$. Since f is continuous on a compact set, it is uniformly continuous. Hence, let n be such that $|f(x) - f(y)| < \epsilon/2$ whenever $|x - y| \le 1/n$. By the density of $\mathbb{Q}$ in $\mathbb{R}$, we can choose $\phi \in \mathcal{C}_n$ such that $|\phi(k/n) - f(k/n)| < \epsilon/2$ for any k.

Now, let $x \in [0,1]$. Then x falls between $x_k = k/n$ and $x_{k+1} = (k+1)/n$ for some k. Hence, there exists a $\lambda \in [0,1]$ such that $x = \lambda x_k + (1 - \lambda) x_{k-1}$. By linearity of ϕ on $[x_k, x_{k+1}]$ and convexity of the absolute value, we have that

$$\begin{aligned} |\phi(x) - f(x)| &= |\lambda(\phi(x_k) - f(x)) + (1 - \lambda)(\phi(x_{k+1}) - f(x))| && (5)\\ &\le \lambda |\phi(x_k) - f(x)| + (1 - \lambda) |\phi(x_{k+1}) - f(x)|. && (6) \end{aligned}$$

But notice that since $|x_k - x| \leq 1/n$, uniform continuity of f yields that

$$|\phi(x_k) - f(x)| \leq |\phi(x_k) - f(x_k)| + |f(x_k) + f(x)| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{7}$$

Reasoning similarly for x_{k+1}, eq. 6 becomes

$$|\phi(x) - f(x)| \leq \lambda\epsilon + (1-\lambda)\epsilon = \epsilon \tag{8}$$

This inequality holds uniformly in x, so $d(\phi, f) \leq \epsilon$, and so we conclude that $\mathcal{D}$ is dense in $C([0,1])$. Therefore, any set that is open in $C([0,1])$ is a countable union of open balls, and so in turn we conclude that the open balls generate $\mathcal{B}$.

But now, for any open ball $B_\epsilon(\omega')$, we have that

$$\begin{aligned} B_\epsilon(\omega') &= \left\{\omega : \sup_{t\in[0,1]} |\omega(t) - \omega'(t)| < \epsilon\right\} && (9)\\ &= \left\{\omega : \sup_{q\in[0,1]\cap\mathbb{Q}} |\omega(q) - \omega'(q)| < \epsilon\right\} && (10)\\ &= \bigcup_{n=1}^{\infty} \bigcap_{q\in[0,1]\cap\mathbb{Q}} \{\omega : |X_q(\omega) - \omega'(q)| \leq \epsilon - 1/n\}. && (11) \end{aligned}$$

But this is a countable union of $\mathcal{F}$-measurable sets and so it itself $\mathcal{F}$-meansurable. Hence, $\mathcal{B} \subseteq \mathcal{F}$.

Problem 3

(a) We have that

$$\{X_{N_k} \leq b\} = \bigcup_{n\in\overline{\mathbb{N}}} \{X_{N_k} \leq b, N_k = n\} = \bigcup_{n\in\overline{\mathbb{N}}} (\{X_n \leq b\} \cap \{N_k = n\}) \tag{12}$$

This is a countable union of measurable sets. Hence, each $\{Z_k \leq b\}$ is measurable and so each Z_k is a random variable.

(b) By almost sure convergence of the sequences, let A and B be probability-1 events such that on A, $X_n \to X_\infty$ and on B, $N_k \to \infty$. Then $A \cap B$ is an intersection of probability-1 events, and so itself has probability 1.

Moreover, on this event, $X_{a_k} \to X_\infty$ along any sequence a_k diverging to infinity. But on this event, N_k is such a sequence, so we have that $X_{N_k} \to X_\infty$ on a probability-1 event. That is, X_{N_k} converges almost surely to X_∞.

(c) Consider the sequence of random variables X_n on $[0,1)$ with the uniform measure defined by

$$X_{2^k+\ell} = \Big\{\mathbb{I}_{[\ell 2^{-k},(\ell+1)2^{-k})} \quad \text{if } \ell = 0, \ldots, 2^{k-1}. \tag{13}$$

We have that $\mathbf{P}(|X_{2^k+\ell}| > \epsilon) = 2^{-k}$ so that $X_n \xrightarrow{\mathbf{P}} X_\infty := 0$. However, notice that for any k,

$$\sum_{\ell=0}^{2^k-1} X_{2^k+\ell} = \mathbb{I}_{[0,1)}. \tag{14}$$

Therefore, with probability 1, for any n, exactly one of $\{X_{2^k}, \dots, X_{2^{k+1}-1}\}$ is equal to 1, while the rest are 0. In particular, if we define (taking $\min \varnothing = \infty$)

$$N_k = \min\{n \in \{2^k, \dots, 2^{k+1} - 1\} : X_n = 1\}, \tag{15}$$

then surely, $N_k \geq 2^k$ so that $N_k \to \infty$ almost surely, and so in probability, while also $X_{N_k} = 1$ almost surely and so $X_{N_k} \xrightarrow{\mathbf{P}} 1$.

(d) We use the same counter-example as in the previous question, noting that $\mathbb{E}|X_{2^k+\ell}| = 2^{-k} \to 0$ and $N_k \xrightarrow{\text{a.s.}} \infty$ but that $\mathbb{E}|X_{N_k} - 1| = 0$ so that $X_{N_k} \xrightarrow{L_1} 1$.

Problem 4

(a) For $\Omega_M = [-M, M]^d$, define the family of sets

$$\mathcal{G}_M = \Big\{A \in \mathcal{B}_{\mathbb{R}^d} : \int f_{\Omega_M}(x)\mathbb{I}_A(x)\,\mathrm{d}x = 0\Big\}. \tag{16}$$

To see that this is a λ-system, first notice that $\Omega_M = [-M, M]^d \in \mathcal{R}$ and so $\mathbb{R}^d \in \mathcal{G}_M$ let $A, B \in \mathcal{G}_M$ such that $A \subseteq B$. We then have that

$$\int f_{\Omega_M}\mathbb{I}_{B\setminus A}\,\mathrm{d}x = \int f_{\Omega_M}\mathbb{I}_B\,\mathrm{d}x - \int f_{\Omega_M}\mathbb{I}_A\,\mathrm{d}x = 0, \tag{17}$$

so that $B \setminus A \in \mathcal{G}$.

Next, let $A_1 \subseteq A_2 \subseteq \cdots$ be a sequence in $\mathcal{G}_M$ increasing to A. Since $f_{\Omega_M} \in L_1$ and $|f_{\Omega_M}|\mathbb{I}_{A_n} \leq |f_{\Omega_M}|\mathbb{I}_A$, the dominated convergence theorem thus yields that

$$\int f_{\Omega_M}\mathbb{I}_A\,\mathrm{d}x = \lim_{n\to\infty}\int f_{\Omega_M}\mathbb{I}_{A_n}\,\mathrm{d}x = 0. \tag{18}$$

Hence, $\mathcal{G}_M$ is closed under increasing limits and so is a λ-system. Since $\mathcal{R}\} \subseteq \mathcal{G}_M$ is a π-system, Dynkin's π-λ theorem then yields that

$$\mathcal{B}_{\mathbb{R}^d} = \sigma(\mathcal{R}) \subseteq \mathcal{G}_M. \tag{19}$$

In particular, $\{x : f_{\Omega_M}(x) > 0\}, \{x : f_{\Omega_M}(x) < 0\} \in \mathcal{G}_M$ so that

$$\int |f_{\Omega_M}|\,\mathrm{d}x = \int f_{\Omega_M}\mathbb{I}_{\{f_{\Omega_M}>0\}}\,\mathrm{d}x - \int f\mathbb{I}_{\{f_{\Omega_M}<0\}}\,\mathrm{d}x. = 0 \tag{20}$$

Therefore, $f_{\Omega_M} = 0$ almost everywhere for any M. But notice that $\{f_{\Omega_M} \neq 0\}$ is an sequence of sets increasing to $\{f = 0\}$, so continuity of measure yields that $f = 0$ almost everywhere.

(b) For any $R = (a_1, b_1) \times \cdots \times (a_n, b_n)$, define the functions $\eta_\epsilon^R \in C_c^\infty(\mathbb{R}^n)$ by

$$\eta_\epsilon^R(t_1, \dots, t_n) = \prod_{i=1}^n \eta_\epsilon\Big(\frac{2(t_i - a_i)}{b_i - a_i} - 1\Big). \tag{21}$$

In particular, we have that that $\eta_\epsilon^R \to \mathbb{I}_R$ pointwise (though there is no guarantee that this convergence is monotone). Moreover, $|f\eta_\epsilon^R| = |f|\eta_\epsilon^R \leq |f|\mathbb{I}_{\overline{R}} = f_{\overline{R}}$, where the latter function is integrable since $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $\overline{R}$ is compact. Hence, the dominated convergence yields that

$$\int f\mathbb{I}_R \, \mathrm{d}x = \lim_{\epsilon\to 0} \int f\eta_\epsilon^R \, \mathrm{d}x = 0. \tag{22}$$

Since this holds for any rectangle R, we have from part (a) that $f = 0$ almost everywhere.

Problem 5

(a) We first prove the result for functions of the form

$$\psi = \sum_{k=1}^n c_i \mathbb{I}_{[a_i,b_i)}, \tag{23}$$

which we will call interval-simple functions.

In particular, we have that, as $\epsilon \to 0$,

$$\begin{aligned}
\|\mathsf{T}_\epsilon\psi - \psi\|_p &= \left\| \sum_{i=1}^n c_i(T_\epsilon \mathbb{I}_{[a_i,b_i)} - \mathbb{I}_{[a_i,b_i)}) \right\|_p && (24)\\
&\leq \sum_{i=1}^n c_i \left\| \mathbb{I}_{[a_i+\epsilon,b_i+\epsilon)} - \mathbb{I}_{[a_i,b_i)} \right\|_p && (25)\\
&\leq \sum_{i=1}^n c_i[(a_i+\epsilon-a_i) + (b_i+\epsilon-b_i)]^{1/p} && (26)\\
&= (2\epsilon)^{1/p} \sum_{i=1}^n c_i && (27)\\
&\to 0. && (28)
\end{aligned}$$

To complete the proof, we will show that such functions are dense in L_p.

Let $\phi = \sum_{i=1}^n c_i \mathbb{I}_{B_i}$ be a simple function supported inside the set $[-M, M]$ and let $\epsilon > 0$. Since $\mathcal{A}$, the set of finite unions of fingernail sets, is an algebra that generates $\mathcal{B}$, we can apply Exercise 1.2.15 (a) of Dembo's notes. In particular, for each B_i, there exists a $A_i \in \mathcal{A}$ such that for $\mu = \mathcal{L}|_{[-M,M]}/2M$, the probability measure formed by restricting the Lebesgue measure to $[-M, M]$,

$$\mathcal{L}(A_i \Delta B_i) = 2M\mu(A_i \Delta B_i) < \left(\frac{\epsilon}{nc_i}\right)^p. \tag{29}$$

But then we have that, for $\psi = \sum_{i=1}^n c_i \mathbb{I}_{A_i}$, which is an interval-simple function since each A_i is a finite union of fingernail sets,

$$\begin{aligned}
\|\phi - \psi\|_p &\leq \sum_{i=1}^n c_i \|\mathbb{I}_{B_i} - \mathbb{I}_{A_i}\|_p && (30)\\
&= \sum_{i=1}^n c_i \mathcal{L}(A_i \Delta B_i)^{1/p} && (31)\\
&< \epsilon. && (32)
\end{aligned}$$

Now, let $f \in L_p$ and let $\epsilon > 0$. By the dominated convergence theorem, $f\mathbb{I}_{[-M,M]} \xrightarrow{L_p} f$, so let M be such that $\left\|f - f\mathbb{I}_{[-M,M]}\right\|_p < \epsilon/3$. Next, since simple functions are dense in L_p, let ϕ be a simple function supported on $[-M, M]$ such that $\left\|f\mathbb{I}_{[-M,M]} - \phi\right\|_p < \epsilon/3$. Finally, by the result that we just proved, let ψ be an interval-simple function such that $\|\phi - \psi\|_p < \epsilon/3$. Therefore, we have that

$$\|f - \psi\|_p \leq \left\|f - f\mathbb{I}_{[-M,M]}\right\|_p + \left\|f\mathbb{I}_{[-M,M]} - \phi\right\|_p + \|\phi - \psi\|_p < \epsilon. \tag{33}$$

Notice that for any $f \in L_p$ and interval-simple function ψ, we have that

$$\begin{aligned}
\|\mathsf{T}_\epsilon f - f\|_p &\leq \|\mathsf{T}_\epsilon f - \mathsf{T}_\epsilon \psi\|_p + \|\mathsf{T}_\epsilon \psi - \psi\|_p + \|\psi - f\|_p & (34)\\
&= 2\|f - \psi\|_p + \|\mathsf{T}_\epsilon \psi - \psi\|_p & (35)\\
&\xrightarrow{\epsilon \to 0} 2\|f - \psi\|_p . & (36)
\end{aligned}$$

But since interval-simple functions are dense in L_p, taking the infimum over all such functions yields that $\limsup_{\epsilon \to 0} \|\mathsf{T}_\epsilon f - f\|_p = 0$.

(b) For each x, let $\mathsf{F}_x f$ be the function defined by $\mathsf{F}_x f(y) = f(x - y)$. Notice that $\|\mathsf{F}_x f\|_p = \|f\|_p$

We have by Hölder's inequality that

$$\begin{aligned}
|f \star g(x)| &\leq \int |f(x - y) g(y)| \, \mathrm{d}y & (37)\\
&= \|\mathsf{F}_x f \cdot g\|_1 & (38)\\
&\leq \|\mathsf{F}_x f\|_p \|g\|_q & (39)\\
&= \|f\|_p \|g\|_q . & (40)
\end{aligned}$$

Since the above holds uniformly in x and since $f \in L_p, g \in L_q$, we conclude that

$$\sup_x |f \star g(x)| \leq \|f\|_p \|g\|_q < \infty. \tag{41}$$

(c) Let $x, x' \in \mathbb{R}$. We again use Hölder's inequality to compute

$$\begin{aligned}
|f \star g(x) - f \star g(x')| &\leq \int |f(x - y) - f(x' - y)||g(y)| \, \mathrm{d}y & (42)\\
&\leq \left[\int |f(x - y) - f(x' - y)|^p \, \mathrm{d}y\right]^{1/p} \|g\|_q . & (43)
\end{aligned}$$

Making the substitution $z = x - y$, this becomes

$$\left[\int |f(z) - f(z - (x - x'))|^p \, \mathrm{d}z\right]^{1/p} \|g\|_q = \|f - \mathsf{T}_{x - x'} f\|_p \|g\|_q . \tag{44}$$

By the result of part (a), this converges to 0 as $x - x' \to 0$ at a rate depending on x and x' only through their difference. Therefore, $f \star g$ is uniformly continuous.

(d) Notice that $\star$ is bilinear. In particular, if we define $f_M = f\mathbb{I}_{[-M,M]}$ and $g_M = f\mathbb{I}_{[-M,M]}$, then we have from part (b) that

$$
\begin{aligned}
|f \star g(x)| &\leq |f_M \star g_M(x)| + |(f - f_M) \star g(x)| + |f_M \star (g - g_M)(x)| && (45)\\
&\leq |f_M \star g_M(x)| + \|f - f_M\|_p \|g\|_q + \|f_M\|_p \|g - g_M\|_q && (46)\\
&\leq |f_M \star g_M(x)| + \|f - f_M\|_p \|g\|_q + \|f\|_p \|g - g_M\|_q . && (47)
\end{aligned}
$$

But notice that when $|x| > 2M$, the sets $[-M, M]$ and $[x - M, x + M]$ are disjoint, so we have that

$$
\begin{aligned}
f_M \star g_M(x) &= \int f(x-y)g(y)\mathbb{I}_{[-M,M]}(x-y)\mathbb{I}_{[-M,M]}(y)\,\mathrm{d}y && (48)\\
&= \int f(x-y)g(y)\mathbb{I}_{[-M,M]\cap[x-M,x+M]}(y)\,\mathrm{d}y && (49)\\
&= 0. && (50)
\end{aligned}
$$

Therefore, we see that for any M,

$$
\limsup_{|x|\to\infty} |f \star g(x)| \leq \|f - f_M\|_p \|g\|_q + \|f\|_p \|g - g_M\|_q . \quad (51)
$$

Taking $M \to \infty$, the dominated convergence theorem yields that the right-hand side converges to 0. Hence, we conclude that $\lim_{|x|\to\infty} f \star g(x) = 0$.