Stat 310A/Math 230A Theory of Probability

Midterm

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Solutions should be complete and concisely written. Please use a separate booklet for each problem.

You have 1.5 hours but you are not required to solve all the problems!!!

Just solve those that you can solve within the time limit.

You can consult any of the listed textbooks (Dembo, Billingsley, Durrett, Williams) and your notes, but nothing else. You cannot use computers or other devices. You can cite theorems (propositions, corollaries, lemmas, etc.) from Dembo's lecture notes by number, and exercises you have done as homework by number as well. Any other non-elementary statement must be proven!

Problem 1

[20 points]

Let $f : \mathbb{R} \to \mathbb{R}$ be the sine function: $f(x) = \sin(x)$.

(*a*) Let $\mathcal{B}$ be the Borel σ-algebra on $\mathbb{R}$ and $\mathcal{F} = \sigma(f)$ (the σ-algebra generated by f, i.e. the smallest σ algebra such that $f : (\mathbb{R}, \mathcal{F}) \to (\mathbb{R}, \mathcal{B})$ is measurable). Is $\mathcal{F} = \mathcal{B}$? Prove your answer.

(*b*) Is $g(x) = x$ measurable on this σ-algebra $\mathcal{F}$? Prove your answer.

Problem 2

[20 points]

Before starting, it might be useful to recall the following classical result.

Theorem 1 (Stone-Weierstrass)**.** *Let $f : K \to \mathbb{R}$ be a continuous function on a compact set $K \subset \mathbb{R}^n$. Then there exists a sequence of polynomials $p_n(x)$ such that $\lim_{n\to\infty} \sup_{x\in K} |p_n(x) - f(x)| = 0$.*

(*a*) Let $\Omega = [a_1, b_1] \times \cdots \times [a_n, b_n]$ be a bounded rectangle in $\mathbb{R}^n$, and μ_1, μ_2 be two finite Borel measures on Ω. Assume that for any $k_1, \dots k_n \in \mathbb{N} = \{0, 1, 2, \dots\}$, we have (here $\boldsymbol{x} = (x_1, \dots, x_n)$ is used to denote a vector in $\mathbb{R}^n$)

$$\int_\Omega x_1^{k_1} \cdots x_n^{k_n} \mu_1(\mathrm{d}\boldsymbol{x}) = \int_\Omega x_1^{k_1} \cdots x_n^{k_n} \mu_2(\mathrm{d}\boldsymbol{x}) \,. \tag{1}$$

Prove that $\mu_1 = \mu_2$.

(*b*) Now, let $X_1, X_2, \dots, X_n$ be random variables such that, for some $M > 0$, $\mathbb{P}(|X_i| > M) = 0$ for all $i \le n$, and

$$\mathbb{E}\{X_1^{k_1} X_2^{k_2} \cdots X_n^{k_n}\} = \mathbb{E}\{X_1^{k_1}\}\mathbb{E}\{X_2^{k_2}\} \cdots \mathbb{E}\{X_n^{k_n}\} \tag{2}$$

for all $k_1, k_2, \dots, k_n \in \mathbb{N}$.

Prove that $X_1, X_2, \dots, X_n$ are mutually independent.

[Hint: The setting of part (*b*) is independent of part (*a*), but you can use the result of part (*a*) to solve part (*b*).]

(c) Let $\boldsymbol{X} = (X_1, X_2, \dots, X_n)$ be a random vector such that, for some $M > 0$, $\mathbb{P}(|X_i| > M) = 0$ for all $i \le n$. Assume there exists a neighborhood U of the origin in $\mathbb{R}^n$ such that

$$\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in U \quad \Rightarrow \quad \mathbb{E}\, e^{\langle \boldsymbol{\lambda}, \boldsymbol{X} \rangle} = \prod_{i=1}^{n} F_i(\lambda_i)\,, \tag{3}$$

for some functions $F_i : \mathbb{R} \to \mathbb{R}$. Prove that $X_1, \dots, X_n$ are mutually independent.

Problem 3

[20 points]

Let $(Z_n)_{n \ge 1}$ be a sequence of i.i.d. random variables uniformly distributed on the interval $[0, 1]$. For $n \ge 1$, let $X_{n,1}, X_{n,2}, \dots, X_{n,n}$ be random variables defined recursively by setting $X_{1,1} = Z_1$ and for $n \ge 2$,

$$X_{n,n} = Z_n^{1/n}\,, \tag{4}$$

$$X_{n,k} = X_{n-1,k} \cdot Z_n^{1/n} \quad \text{for} \quad k = 1, 2, \dots, n-1. \tag{5}$$

Denoting by P_n the joint law of $\boldsymbol{X}_n = (X_{n,1}, X_{n,2}, \dots, X_{n,n})$, prove that this has density (with respect to the Lebesgue measure on $\mathbb{R}^n$):

$$f_n(x_1, x_2, \dots, x_n) = n!\, \mathbf{1}(0 < x_1 < x_2 < \cdots < x_n < 1)\,. \tag{6}$$