STAT 310A Midterm 2026: Solutions

Problem 1

Let $f(x) = \sin x$ and $\mathcal{F} = \sigma(f)$.

(a) Every set in $\mathcal{F}$ must be such that. if $f(x) = f(y)$, then $x \in A$ if and only if $y \in A$ for every $A \in \mathcal{F}$. Indeed, this property holds for every set $f^{-1}(B)$, $B \in \mathcal{B}(\mathbb{R})$, and the collection of sets with this property is a σ-algebra. Since $f(0) = f(2\pi)$, the Borel set $\{0\}$ does not have this property and hence is not in $\mathcal{F}$.

(b) The identity map $g(x) = x$ is not $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable. If it were, then $g^{-1}(\{0\}) = \{0\}$ would belong to $\mathcal{F}$, contradicting part (a).

Problem 2

(a) The states assumption implies (by linearity of integral) that $\mu_1(p) = \mu_2(p)$ for every polynomial $p : \mathbb{R}^n \to \mathbb{R}$. For any continuous $h : \Omega \to \mathbb{R}$, choose polynomials p_j converging uniformly to h on Ω (these exist by the Weierstrass theorem). Since the measures are finite,

$$\left| \int_\Omega (h - p_j)\, d\mu_i \right| \leq \mu_i(\Omega)\, \|h - p_j\|_\infty \longrightarrow 0 \qquad (i = 1, 2).$$

Consequently, $\int h\, d\mu_1 = \int h\, d\mu_2$ for every continuous h.

Now let C be closed in Ω and, when $C \neq \varnothing$, set $h_j(x) = \max\{1 - j\ \mathrm{dist}(x, C), 0\}$. These are continuous functions on Ω, bounded by 1, and $h_j \to \mathbf{1}_C$ pointwise. Dominated convergence gives $\mu_1(C) = \mu_2(C)$; the same equality is immediate for $C = \varnothing$. By Caratheodory's extension theorem $\mu_1 = \mu_2$.

(b) Let ν be the joint law of $(X_1, \ldots, X_n)$ and let ν_i be the law of X_i. Both ν and the product measure $\nu_1 \times \cdots \times \nu_n$ are probability measures on $\Omega = [-M, M]^n$, since all coordinates lie in $[-M, M]$ almost surely. The assumed identity says that their integrals of every monomial $x_1^{k_1} \cdots x_n^{k_n}$ agree. By part (a), $\nu = \nu_1 \otimes \cdots \otimes \nu_n$. Equality of joint law and the product of its marginals is mutual independence.

(c) Write $H(\lambda) = \mathbb{E}e^{\langle \lambda, X \rangle}$ and $H_i(t) = \mathbb{E}e^{tX_i}$. The assumed factorization at $\lambda = 0$ gives $\prod_i F_i(0) = H(0) = 1$; in particular, every $F_i(0)$ is nonzero. Putting all but the ith coordinate of λ equal to zero gives, for λ sufficiently close to zero,

$$H_i(\lambda_i) = F_i(\lambda_i) \prod_{j \neq i} F_j(0).$$

Multiplying these identities and using $\prod_j F_j(0) = 1$ yields

$$H(\lambda) = \prod_{i=1}^n H_i(\lambda_i) \quad \text{in a neighborhood of } 0.$$

The boundedness of X permits differentiation under the expectation any number of times there: each differentiated integrand is bounded by a constant depending on the multi-index and the neighborhood. Thus, for every $(k_1, \ldots, k_n) \in \mathbb{N}_0^n$,

$$\mathbb{E} \prod_{i=1}^{n} X_i^{k_i} = \partial_1^{k_1} \cdots \partial_n^{k_n} H(\lambda)\Big|_{\lambda=0} = \prod_{i=1}^{n} H_i^{(k_i)}(0) = \prod_{i=1}^{n} \mathbb{E} X_i^{k_i}.$$

Part (b) now proves that $X_1, \ldots, X_n$ are mutually independent.

Problem 3

We prove the density formula by induction. For $n = 1$, $X_{1,1} = Z_1$ has density $\mathbf{1}_{(0,1)}$ (the endpoints have probability zero).

Suppose that $\boldsymbol{X}_{n-1}$ has the stated density. Put $U = Z_n^{1/n}$. As Z_n is independent of $\boldsymbol{X}_{n-1}$, so is U; moreover, for $0 < u < 1$,

$$\mathbb{P}(U \le u) = u^n, \qquad f_U(u) = n u^{n-1}.$$

The recursion is the transformation

$$(y_1, \ldots, y_{n-1}, u) \longmapsto (x_1, \ldots, x_n) = (u y_1, \ldots, u y_{n-1}, u).$$

It maps $0 < y_1 < \cdots < y_{n-1} < 1$, $0 < u < 1$ bijectively onto $0 < x_1 < \cdots < x_n < 1$. Its Jacobian determinant is u^{n-1}, and its inverse has $y_k = x_k / x_n$ and $u = x_n$. Hence the change-of-variables formula and the induction hypothesis give, on this ordered region,

$$f_n(x_1, \ldots, x_n) = \frac{f_{n-1}(x_1/x_n, \ldots, x_{n-1}/x_n) f_U(x_n)}{x_n^{n-1}} = \frac{(n-1)!\, n x_n^{n-1}}{x_n^{n-1}} = n!.$$

Outside the ordered region the density is zero. Therefore

$$f_n(x_1, \ldots, x_n) = n!\, \mathbf{1}\{0 < x_1 < \cdots < x_n < 1\}.$$