Stats 310A Session 1

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1 Constructing the uniform measure on $[0, 1)$

In this section, we briefly describe how to construct the uniform measure (also the Lebesgue measure) on $[0, 1)$. We will mostly follow Dembo's notes and Sections 1, 2 in Billingsley [1].

Consider sets that are finite disjoint unions of intervals in $[0, 1)$. Let $\mathcal{B}_0$ denote this family of sets:

$$\mathcal{B}_0 = \left\{ A = \bigcup_{k=1}^{n} [a_k, b_k) : 0 \le a_1 < b_1 < \cdots < a_n < b_n \le 1, n \in \mathbb{N} \right\}.$$

It is easy to verify that $\mathcal{B}_0$ is an *algebra*: it is closed under complement and union, and $\emptyset \in \mathcal{B}_0$.

Now, define set function $\lambda : \mathcal{B}_0 \to [0, 1]$ as

$$\lambda(A) = \sum_{k=1}^{n} (b_k - a_k) \quad \text{for} \quad A = \bigcup_{k=1}^{n} [a_k, b_k).$$

We claim that λ is a probability measure on $\mathcal{B}_0$. Clearly, $\lambda(A) \in [0, 1]$ for all $A \in \mathcal{B}_0$, $\lambda(\emptyset) = 0$, and $\lambda([0, 1)) = 1$. It remains to show that λ is *countably additive*, that is,

$$A = \bigcup_{k=1}^{\infty} A_k, \; A, A_k \in \mathcal{B}_0, \; A_k \text{ disjoint} \;\; \text{implies} \;\; \lambda(A) = \sum_{i=1}^{\infty} \lambda(A_i).$$

To achieve this, we need the following result on the length of intervals. For a (finite) interval $I = [a, b)$, let $|I| = b - a$ denote its length.

Lemma 1.1 (Theorem 1.3, [1]). *Let I and $\{I_k\}_{k=0}^{\infty}$ be intervals.*

(i) If $\bigcup_k I_k \subset I$ and the I_k are disjoint, then $\sum_k |I_k| \le |I|$.

(ii) If $I \subset \bigcup_k I_k$, then $|I| \le \sum_k |I_k|$.

(iii) If $I = \bigcup_k I_k$ and the I_k are disjoint, then $|I| = \sum_k |I_k|$.

In particular, (iii) ensures that the length of an interval is not only finitely but also countably additive, which we will now use to show that λ is also countably additive. Let $A = \bigcup_{k=1}^{n} I_k$ and $A_k = \bigcup_{j=1}^{m_k} J_{kj}$ be the disjoint interval representations. Then for all i, we have

$$I_i = I_i \cap A = I_i \bigcap \left(\bigcup_{k=1}^{\infty} \bigcup_{j=1}^{m_k} J_{kj} \right) = \bigcup_{k=1}^{\infty} \bigcup_{j=1}^{m_k} I_i \cap J_{kj}.$$

$I_i \cap J_{kj}$ are disjoint intervals, so we can apply Lemma 1.1(iii) twice to get

$$\lambda(A) = \sum_{i=1}^{n} |I_i| = \sum_{i=1}^{n}\sum_{k=1}^{\infty}\sum_{j=1}^{m_k} |I_i \cap J_{kj}| = \sum_{k=1}^{\infty}\sum_{j=1}^{m_k} |J_{kj}| = \sum_{k=1}^{\infty} \lambda(A_k).$$

This completes the proof.

As λ is a probability measure on the algebra $\mathcal{B}_0$, the Caratheodory extension theorem states that λ has an unique extension onto $\mathcal{B} = \sigma(\mathcal{B}_0)$, giving the Lebesgue measure on Borel sets. Note that λ can be extended onto $\mathcal{G}$, the family of measurable sets, which is strictly larger than $\mathcal{B}$.

Proof of Lemma 1.1 Let $I = [a, b)$ and $I_k = [a_k, b_k)$.

(i) *Finite case.* Suppose there are n intervals. We perform induction on n. The result is obvious when $n = 1$. Assume the result is true for $n-1$, and let I_k be sorted in the increasing order, then we have $b_k \le a_n < b_n \le b$ for all $k \le n-1$. Now, the smaller interval $[a, a_n)$ contains $\bigcup_{k=0}^{n-1} I_k$, so by the inductive assumption we have $\sum_{k=1}^{n-1} |I_k| \le a_n - a$. This gives

$$\sum_{k=1}^{n} |I_k| = \sum_{k=1}^{n-1} |I_k| + (b_n - a_n) \le (a_n - a) + (b_n - a_n) = b_n - a \le b - a = |I|,$$

verifying the result for n.

Infinite case. For all n we have $\sum_{k=1}^{n} |I_k| \le |I|$ by the finite case. Letting $n \to \infty$ gives the result.

(ii) *Finite case.* Induction. Assume the result is true for $n-1$. Then, there is at least one interval, WLOG $[a_n, b_n)$, such that $a_n < b \le b_n$. This interval covers the $[a_n, b)$ portion of I, so the rest must cover $[a, a_n)$. By the inductive assumption, $a_n - a \le \sum_{k=1}^{n-1} |I_k|$. This gives

$$|I| = b - a = (a_n - a) + (b - a_n) \le \sum_{k=1}^{n-1} |I_k| + (b_n - a_n) = \sum_{k=1}^{n} |I_k|.$$

Infinite case. By the assumption

$$[a, b) \subset \bigcup_{k=1}^{\infty} [a_k, b_k),$$

we have

$$[a, b - \varepsilon] \subset \bigcup_{k=1}^{\infty} \left(a_k - \frac{\varepsilon}{2^k}, b_k\right) \quad \text{for all } 0 < \varepsilon < b - a,$$

as the LHS is a smaller set and the RHS is a larger set. However, as the interval $[a, b - \varepsilon]$ is compact and the RHS is an open cover, it must have a finite subcover, WLOG $k \in \{1, \dots, n\}$, giving that

$$\bigcup_{k=1}^{n} \left(a_k - \frac{\varepsilon}{2^k}, b_k\right) \supset [a, b - \varepsilon] \supset [a, b - \varepsilon).$$

Applying the finite case, we get

$$b - a \le \varepsilon + \sum_{k=1}^{n} \left(b_k - a_k + \frac{\varepsilon}{2^k}\right) \le \sum_{k=1}^{n} (b_k - a_k) + 2\varepsilon \le \sum_{k=1}^{\infty} (b_k - a_k) + 2\varepsilon.$$

Taking $\varepsilon \to 0$ gives the desired result.

(iii) Follows from (i) and (ii).

□

2 Proof of existence in Caratheodory's extension theorem

In this section, we prove the existence part in Caratheodory's extension theorem: a probability measure P on a field $\mathcal{F}_0$ has an extension to $\sigma(\mathcal{F}_0)$. We will follow Section 3 in Billingsley [1].

For any set $A \subset \Omega$, define its *outer measure* by

$$P^*(A) = \inf_{\{A_n\} \text{ covers } A} \sum_{n=1}^{\infty} P(A_n).$$

$P^*(A)$ measures the size of a set $\mathcal{A}$ by its smallest countable $\mathcal{F}_0$-cover. One can check that it satisfies the following properties:

(i) $P^*(\emptyset) = 0$.

(ii) Nonnegativity: $P^*(A) \geq 0$ for all $A \subset \Omega$.

(iii) Monotonicity: $A \subset B$ implies $P^*(A) \leq P^*(B)$.

(iv) Countable subadditivity: if $A \subset \bigcup_n A_n$, then $P^*(A) \leq \sum_n P^*(A_n)$.

Properties (i) - (iii) are relatively easy to verify; (iv) can be verified by constructing covers of A_n wiithin $\varepsilon/2^n$ of the outer measure. We note that (iv) also implies finite subadditivity, in particular, $P^*(A \cup B) \leq P^*(A) + P^*(B)$.

Now, we define a class of sets

$$\mathcal{G} := \{A \subset \Omega : \mathbb{P}^*(E \cap A) + P^*(E \cap A^c) = P^*(E) \text{ for all } E \subset \Omega\}.$$

Our goal is to show that P^* restricted on $\mathcal{G}$ is the extension of P. The class $\mathcal{G}$ contains $\sigma(\mathcal{F}_0)$ and is what we will later call *measurable sets*. Also, as a consequence of finite subadditivity, the "$\geq$" direction in the defining equality always holds, so we only need to check the "$\leq$" direction in order to show that a set is in $\mathcal{G}$.

Lemma 2.1. *The class $\mathcal{G}$ is an algebra.*

Proof Clearly $\emptyset \in \mathcal{G}$, and $A \in \mathcal{G}$ implies $\mathcal{A}^c \in \mathcal{G}$ by symmetry of the definition, so it remains to show that $\mathcal{G}$ is closed under intersection. For any $A, B \in \mathcal{G}$ and $E \subset \Omega$, we have

$$\begin{aligned} P^*(E) &= P^*(E \cap A) + P^*(E \cap A^c) \\ &= P^*(E \cap A \cap B) + P^*(E \cap A \cap B^c) + P^*(E \cap A^c \cap B) + P^*(E \cap A^c \cap B^c) \\ &\geq P^*(E \cap A \cap B) + P^*\left(E \cap A \cap B^c \bigcup E \cap A^c \cap B \bigcup E \cap A^c \cap B^c\right) \\ &= P^*(E \cap (A \cap B)) + P^*\left(E \cap (A \cap B)^c\right), \end{aligned}$$

which shows $A \cap B \in \mathcal{G}$. □

Lemma 2.2. *If $A_1, A_2, \ldots$ are disjoint $\mathcal{G}$-sets, then for all $E \subset \Omega$,*

$$P^*\left(E \bigcap \left(\bigcup_n A_n\right)\right) = \sum_n P^*(E \cap A_n).$$

Proof *Finite case.* Suppose there are n sets $A_1, \ldots, A_n$. When $n = 1$ this is obvious. Assume the result holds with $n-1$, then letting $B_k = \bigcup_{i=1}^k A_i$ for all k, we have

$$\begin{aligned} P^*(E \cap B_n) &= P^*(E \cap B_n \cap B_{n-1}) + P^*(E \cap B_n \cap B_{n-1}^c) \\ &= P^*(E \cap B_{n-1}) + P^*(E \cap A_n) = \sum_{i=1}^{n-1} P^*(E \cap A_i) + P^*(E \cap A_n) = \sum_{i=1}^{n} P^*(E \cap A_i). \end{aligned}$$

By induction, the result is true for all finite collections.

Infinite case. As P^* is countably subadditive, we need only show the "$\geq$" direction. By monotonicity and the finite case,

$$P^*\left(E \bigcap \left(\bigcup_i A_i\right)\right) \geq P^*\left(E \bigcap \left(\bigcup_{i=1}^n A_i\right)\right) = \sum_{i=1}^n P^*(E \cap A_n)$$

for all n. Letting $n \to \infty$, we get

$$P^*\left(E \bigcap \left(\bigcup_i A_i\right)\right) \geq \sum_{i=1}^\infty P^*(E \cap A_n).$$

□

Lemma 2.3. *The class $\mathcal{G}$ is a σ-algebra, and P^* restricted on $\mathcal{G}$ is countably additive.*

Proof Suppose $A_1, A_2, \ldots$ are disjoint $\mathcal{G}$-sets. Let $B = \bigcup A_i$ and $B_n = \bigcup_{i=1}^n A_i$. For any $E \subset \Omega$, we have

$$P^*(E) = P^*(E \cap B_n) + P^*(E \cap B_n^c) = \sum_{i=1}^n P^*(E \cap A_i) + P^*(E \cap B_n^c) \geq \sum_{i=1}^n P^*(E \cap A_i) + P^*(E \cap B^c).$$

Letting $n \to \infty$, we obtain

$$P^*(E) \geq \sum_{i=1}^\infty P^*(E \cap A_i) + P^*(E \cap B^c) = P^*(E \cap B) + P^*(E \cap B^c),$$

where we applied Lemma 2.2 to get the last equality. This shows that $B \in \mathcal{G}$, and so $\mathcal{G}$ is a σ-algebra. Taking $E = B$ in the above inequality gives countable additivity. □

Lemma 2.4. *We have $\mathcal{F}_0 \in \mathcal{G}$.*

Proof Let $A \in \mathcal{F}_0$ and take any $E \subset \Omega$. For any $\varepsilon > 0$, there exists a cover $\{A_n\} \subset \mathcal{F}_0$ of E such that $\sum_n P(A_n) \leq P^*(E) + \varepsilon$. Let $B_n = A_n \cap A$ and $C_n = A_n \cap A^c$, these sets are all in $\mathcal{F}_0$ and cover $E \cap A$ and $E \cap A^c$, respectively. So we have

$$P^*(E \cap A) + P^*(E \cap A^c) \leq \sum_n P(A_n \cap A) + \sum_n P(A_n \cap A^c) = \sum_n P(A_n) \leq P^*(E) + \varepsilon.$$

Letting $\varepsilon \to 0$, we get $A \in \mathcal{G}$. □

Lemma 2.5. *P^* restricted on $\mathcal{F}_0$ is equal to P, i.e.*

$$P^*(A) = P(A), \text{ for all } A \in \mathcal{F}_0.$$

Proof Let $A \in \mathcal{F}_0$. Clearly A itself covers A, so $P^*(A) \leq P(A)$. Conversely, if $\{A_n\}$ is a $\mathcal{F}_0$-cover of A, then by the countable subadditivity and monotonicity of P on $\mathcal{F}_0$, we have

$$P(A) \leq \sum_n P(A \cap A_n) \leq \sum_n P(A_n).$$

Taking inf over all covers gives that $P(A) \leq P^*(A)$. □

Proof of existence of extension By Lemmas 2.3, 2.4, and 2.5, the outer measure P^* extends P onto $\mathcal{G}$, which is a σ-algebra that contains $\mathcal{F}_0$. Thus, $\mathcal{G} \supset \sigma(\mathcal{F}_0)$. As P^* is a probability measure on $\mathcal{G}$, it is also a probability measure when restricted to $\sigma(\mathcal{F}_0)$. □

References

[1] P. Billingsley. *Probability and measure.* John Wiley & Sons, 2008.