Stats 310A Session 2

October 4, 2019

1 Characterization of distribution functions

In this section, we give necessary and sufficient conditions for a function $F : \mathbb{R} \to [0, 1]$ to be a distribution function.

Theorem 1 (Thm 1.2.36, Dembo's Notes)**.** *A function $F : \mathbb{R} \to [0, 1]$ is a distribution function of some R.V. if and only if*

(a) F is non-decreasing;

(b) $\lim_{x\to\infty} F(x) = 1$ and $\lim_{x\to-\infty} F(x) = 0$;

(c) F is right-continuous, i.e. $\lim_{y\downarrow x} F(y) = F(x)$.

Proof "$\Rightarrow$". Let F be the distribution of some random variable X on a probability space $(\Omega, \mathcal{F}, P)$. Let $x \le y$, then $\{\omega : X(\omega) \le x\} \subseteq \{\omega : X(\omega) \le y\}$, hence $F(x) = \mathbb{P}(X \le x) \le \mathbb{P}(X \le y) = F(y)$. By continuity of P, we have

$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty} P(\{\omega : X(\omega) \le x\}) = P(\lim_{x\to\infty} \{\omega : X(\omega) \le x\}) = P(\Omega) = 1$$

and similarly $\lim_{x\to 0} F(x) = P(\emptyset) = 0$. Take $x \in \mathbb{R}$, we have

$$\lim_{y\downarrow x} \{\omega : X(\omega) \le y\} = \{\omega : X(\omega) \le x\}.$$

Hence by continuity of P,

$$\lim_{y\downarrow x} F(y) = \lim_{y\downarrow x} P(\{\omega : X(\omega) \le y\}) = P(\{\omega : X(\omega) \le x\}) = F(x).$$

"$\Leftarrow$". We define $X^-(\omega) = \sup\{y : F(y) < \omega\}$ on the probability space $((0, 1], \mathcal{B}_{(0,1]}, U)$, i.e. $(0, 1]$ with the uniform distribution. Note that for all $\omega \in (0, 1)$, as F is non-decreasing and its range contains $(0, 1)$, the set $\{y : F(y) \le \omega\}$ is non-empty and has a finite upper bound. Hence $X^- : (0, 1) \to \mathbb{R}$ is well-defined.

We are going to show that the distribution function of X^- equals to F. We claim that for all $x \in \mathbb{R}$,

$$\{\omega : X^-(\omega) \le x\} = \{\omega : \omega \le F(x)\}. \tag{1}$$

This implies that the LHS is in $\mathcal{B}_{(0,1]}$ and that

$$U(\{\omega : X^-(\omega) \le x\}) = U(\{\omega : \omega \le F(x)\}) = U((0, F(x)]) = F(x),$$

so the distribution function of $X^-(\omega)$ is F.

It remains to show (1). Suppose $F(x) \geq \omega$, then by monotonicity $x \geq y$ for all y such that $F(y) < \omega$, giving that $X^-(\omega) = \sup\{y : F(y) < \omega\} \leq x$. Conversely, suppose $X^-(\omega) \leq x$, we claim that $F(x) \geq \omega$ has to be true. If not, then $F(x) < \omega$. By the right continuity of F, there exists some $\varepsilon > 0$ such that $F(x+\varepsilon) < \omega$, giving that

$$X^-(\omega) = \sup\{y : F(y) < \omega\} \geq x + \varepsilon > x,$$

a contradiction. Hence, we must have $F(x) \geq \omega$. □

2 Completion of measure spaces

A nice property about the Lebesgue measure is the following: any subset of a measure-zero set is measurable. To see this, for example on $\mathbb{R}$, let A have measure zero and $B \subset A$. For any $E \subset \mathbb{R}$, we have

$$\begin{aligned} &P^*(E \cap B) + P^*(E \cap B^c) \\ &\leq P^*(E \cap A) + P^*(E \cap B^c \cap A^c) + P^*(E \cap B^c \cap A) \\ &= P^*(E \cap A) + P^*(E \cap A^c) + P^*(E \cap (A \setminus B)) = P^*(E \cap A) + P^*(E \cap A^c) = P^*(E). \end{aligned} \tag{2}$$

The last equality follows as $E \cap (A \setminus B)$ is a subset of A, so $P^*(E \cap (A \setminus B)) \leq P^*(A) = 0$. Hence, B is measurable, and $P(B) \leq P(A) = 0$.

However, such a property might not be present in a general measure space. We are going present a result saying that one can always slightly enlarge the σ-algebra and extend the measure to get this property.

Definition 1 (Def 1.1.34, Dembo's Notes)**.** *We say that a measure space $(\Omega, \mathcal{F}, \mu)$ is* complete *if any subset N of any $B \in \mathcal{F}$ with $\mu(B) = 0$ is also in $\mathcal{F}$.*

Theorem 2 (Thm 1.1.35, Dembo's Notes)**.** *Given a measure space $(\Omega, \mathcal{F}, \mu)$, let*

$$\mathcal{N} = \{N : N \subseteq A \textit{ for some } A \in \mathcal{F} \textit{ with } \mu(A) = 0\}$$

denote the collection of μ-null sets. Then, there exists a complete measure space $(\Omega, \overline{\mathcal{F}}, \overline{\mu})$, called the completion *of the measure space $(\Omega, \mathcal{F}, \mu)$, such that $\overline{\mathcal{F}} = \{F \cup N : F \in \mathcal{F}, N \in \mathcal{N}\}$ and $\overline{\mu} = \mu$ on $\mathcal{F}$.*

Intuitively the result is quite expected: we can add all the μ-null sets into $\mathcal{F}$ and let them have measure zero.

Proof We divide the proof into the following steps.

(1) $\overline{\mathcal{F}}$ is a σ-algebra.

Clearly $\emptyset \in \overline{\mathcal{F}}$. Take any $B \in \overline{\mathcal{F}}$, then $B = F \cup N$ with $F \in \mathcal{F}$ and $N \in \mathcal{N}$. In particular, there exists $A \in \mathcal{F}$ such that $\mu(A) = 0$ and $N \subseteq A$. Thus

$$B^c = F^c \cap N^c = ((F^c \cap A) \cap N^c) \cup ((F^c \cap A^c) \cap N^c) = (F^c \cap A^c) \cup (F^c \cap A \cap N^c).$$

As $F^c \cap A^c \in \mathcal{F}$ and $F^c \cap A \cap N^c \subseteq A$, we have $B^c \in \overline{\mathcal{F}}$. For any $\{B_n\} \in \overline{\mathcal{F}}$, let $B_n = F_n \cup N_n$ and $N_n \subseteq A_n$ be their decompositions, then

$$\bigcup_n B_n = \left(\bigcup_n F_n\right) \cup \left(\bigcup_n N_n\right).$$

As $\bigcup_n N_n \subseteq \bigcup_n A_n$ and $\bigcup_n A_n \in \mathcal{F}$ with $\mu(\bigcup_n A_n) = \sum_n \mu(A_n) = 0$, we have $\bigcup_n N_n \in \mathcal{N}$ and thus $\bigcup_n B_n \in \overline{\mathcal{F}}$.

(2) Define $\overline{\mu}(B) = \mu(F)$ for $B = F \cup N$, $F \in \mathcal{F}$, $N \in \mathcal{N}$. $\overline{\mu}$ is well defined.

We need to verify that if B have two decompositions $B = F_1 \cup N_1$ and $B = F_2 \cup N_2$, then $\mu(F_1) = \mu(F_2)$. Indeed, we have

$$F_1 \subseteq F_1 \cup N_1 = B = F_2 \cup N_2 \subseteq F_2 \cup A_2,$$

where $\mu(A_2) = 0$. Hence $\mu(F_1) \le \mu(F_2 \cup A_2) \le \mu(F_2) + \mu(A_2) = \mu(F_2)$. That $\mu(F_2) \le \mu(F_1)$ follows by exchanging the roles of F_1 and F_2.

(3) $\overline{\mu}$ is a measure on $\overline{\mathcal{F}}$ and agrees with μ on $\mathcal{F}$.

Clearly $\overline{\mu}(\emptyset) = 0$. Let $\{B_n\}$ be a sequence of disjoint sets in $\overline{\mathcal{F}}$ with decompositions $B_n = F_n \cup N_n$. As F_n and N_n are all disjoint, we have

$$\overline{\mu}\left(\bigcup_n B_n\right) = \overline{\mu}\left(\bigcup_n F_n \cup \bigcup_n N_n\right) = \mu\left(\bigcup_n F_n\right) = \sum_n \mu(F_n) = \sum_n \overline{\mu}(B_n).$$

Hence $\overline{\mu}$ is countably additive. For any $F \in \mathcal{F}$, $F = F \cup \emptyset$, so $\overline{\mu}(F) = \mu(F)$.

(4) $(\Omega, \overline{\mathcal{F}}, \overline{\mu})$ is complete.

Take any $B \in \overline{\mathcal{F}}$ with $\overline{\mu}(B) = 0$. Then $B = F \cup N$ for some $F \in \mathcal{F}$ and $N \in \mathcal{N}$. We have $\mu(F) = \overline{\mu}(F) \le \overline{\mu}(B) = 0$, so F itself is a measure-zero set in $\mathcal{F}$, hence $B \in \mathcal{N}$. ($A \cup F$ contains B for A containing N.) So if $C \subseteq B$, then $C \in \mathcal{N}$, and thus $C \in \overline{\mathcal{F}}$.

□

Remark Another way of constructing the completion is to look at the outer measure μ^* on μ^*-measurable sets $\mathcal{G}$. One can show that this construction coincides with our construction, in particular, $\mathcal{G} = \overline{\mathcal{F}}$. (See Exercise 3.10(c) in Billingsley [1].)

References

[1] P. Billingsley. *Probability and measure.* John Wiley & Sons, 2008.