Stats 310A Session 3

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1 Comparison of Riemann and Lebesgue integral

It frequently happens that we are required to compute an integral $\int_{(a,b]} f(x)dx$ where f is Lebesgue measurable on $((a,b], \mathcal{G}_{(a,b]}, \lambda)$. How do we compute this integral? Well, we could follow the definition (approximate by simple functions) or use change of variables formula, both still being quite complicated tasks. In practice, however, we often simply compute the Riemannian integral (e.g. by finding the primitive F and computing $F(b) - F(a)$).

We show that any non-negative Riemann integrable function on $(a,b]$ will also be Lebesgue measurable (hence integrable) with coinciding integral values, justifying their relation.

Definition 1. *A function $f : (a,b] \to [0,\infty]$ is* Riemann integrable *with integral $R(f) < \infty$ if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $|\sum_l f(x_l)\lambda(J_l) - R(f)| \le \varepsilon$ for any $x_l \in J_l$ and $\{J_l\}$ a finite partition of $(a,b]$ into disjoint intervals whose length $\lambda(J_l) \le \delta$.*

Proposition 1 (Proposition 1.3.64, Dembo's Notes)**.** *If $f(x)$ is a non-negative Riemann integrable function on an interval $(a,b]$, then it is also Lebesgue measurable on $(a,b]$ and $\lambda(f) = R(f)$.*

Proof For any $\varepsilon > 0$, there exists some $\delta > 0$, such that for all partition $\{J_l\}$ of size $\le \delta$ and any $x_l \in J_l$

$$R(f) - \varepsilon \le \sum_l f(x_l)\lambda(J_l) \le R(f) + \varepsilon.$$

Define $f_*(J) = \inf\{f(x) : x \in J\}$ and $f^*(J) = \sup\{f(x) : x \in J\}$. Varying x_l in the above bound, we see that

$$R(f) - \varepsilon \le \sum_l f_*(J_l)\lambda(J_l) \le \sum_l f^*(J_l)\lambda(J_l) \le R(f) + \varepsilon.$$

Written differently, if we define for any partition Π

$$\ell(\Pi)(x) = \sum_l f_*(J_l)\mathbf{1}\{x \in J_l\}, \quad u(\Pi)(x) = \sum_l f^*(J_l)\mathbf{1}\{x \in J_l\}.$$

then $\ell(\Pi)$ and $u(\Pi)$ are non-negative simple functions with Lebesgue integrals $\sum_l f_*(J_l)\lambda(J_l)$ and $\sum_l f^*(J_l)\lambda(J_l)$. Consequently, as long as Π has size $\le \delta$, we have $R(f) - \varepsilon \le \lambda(\ell(\Pi)) \le \lambda(u(\Pi)) \le R(f) + \varepsilon$.

Let Π_n be the dyadic partition of $(a,b]$ to 2^n intervals of equal length $(b-a)2^{-n}$. For sufficiently large n, $R(f) - \varepsilon \le \lambda(\ell(\Pi_n)) \le \lambda(u(\Pi_n)) \le R(f) + \varepsilon$. Noting that $u(\Pi_n) \ge u(\Pi_{n+1})$ and so they have a pointwise limit $u(\Pi_n) \downarrow u_\infty$ and similarly $\ell(\Pi_n) \uparrow \ell_\infty$, where u_∞, ℓ_∞ are Lebesgue measurable. By the motonicity of Lebesgue's integral,

$$R(f) - \varepsilon \le \liminf_{n\to\infty} \lambda(\ell(\Pi_n)) \le \lambda(\ell_\infty) \le \lambda(u_\infty) \le \limsup_{n\to\infty} \lambda(u(\Pi_n)) \le R(f) + \varepsilon.$$

Letting $\varepsilon \to 0$ gives $\lambda(\ell_\infty) = \lambda(u_\infty) = R(f)$.

Finally, observe that $\ell_\infty(x) \le f(x) \le u_\infty(x)$, and that

$$\{x : f(x) \neq \ell_\infty(x)\} \subseteq \{x : u_\infty(x) > \ell_\infty(x)\},$$

with the latter a measure-zero set (as $u_\infty \ge \ell_\infty$ and $\int (u_\infty - \ell_\infty) dx = 0$). Hence, by the completeness of the Lebesgue measure, $\{x : f(x) \neq \ell_\infty(x)\}$ is also Lebesgue measurable with measure zero, which implies that f is measurable and $\lambda(f) = \lambda(\ell_\infty)$. □

2 Miscellanous Examples

2.1 Set operations

We have seen some set operations in the the last HW (Exercise 1.2.30, Dembo's Notes). Here we make formal some set operations that will be useful later in the class.

As a motivation, let us think of how we could define the limit of sets (assuming a common superset Ω). Recall that we define the limit of an increasing sequence of sets as $\lim_n A_n = \bigcup_n A_n$ and for a decreasing sequence as $\lim_n A_n = \bigcap_n A_n$. Then, for a general non-monotone sequence, we are going to define the upper and lower limits for the sequence via constructing related monotone sequences. For a sequence of sets $\{A_n\}$, we define

$$\liminf_n A_n = \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} A_n = \{\omega : A_n(\omega) \text{ happens for all large } n\},$$

$$\limsup_n A_n = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} A_n = \{\omega : A_n(\omega) \text{ happens infinitely often}\}.$$

It is easy to verify that $\liminf_n A_n \subseteq \limsup_n A_n$, and we say that the limit of A_n exists if $\liminf_n A_n = \limsup_n A_n$.

Let us practice set operations on an example: let $X_1, X_2, \ldots$ be a sequence of R.V.s and X_∞ be an R.V., all defined on some measure space $(\Omega, \mathcal{F})$. Then,

$$\{\omega : X_n(\omega) \to X_\infty(\omega) \text{ as } n \to \infty\} = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \left\{\omega : |X_n(\omega) - X_\infty(\omega)| \le \frac{1}{k}\right\}.$$

As we will see later, this relation is useful for characterizing almost sure convergence, in particular, for showing that $X_n \overset{a.s.}{\to} X_\infty$ implies $X_n \overset{p}{\to} X_\infty$.

To show this, note that $X_n(\omega) \to X_\infty(\omega)$ happens iff for all $k \in \mathbb{N}$, $|X_n(\omega) - X_\infty(\omega)| \le k^{-1}$ for all large n. Hence,

$$\{\omega : X_n(\omega) \to X_\infty(\omega)\} = \bigcap_{k=1}^{\infty} \left\{\omega : |X_n(\omega) - X_\infty(\omega)| \le \frac{1}{k} \text{ for all large } n\right\},$$

which implies the result.

2.2 Generated σ-algebra

The following gives an example in which for an increasing sequence of σ-algebras $\mathcal{F}_n$, $\bigcup_n \mathcal{F}_n$ is a σ-algebra, thereby showing that $\sigma(\bigcup_n \mathcal{F}_n) \supsetneq \bigcup_n \mathcal{F}_n$ in general.

On $\mathbb{R}$, define

$$\mathcal{F}_n = \sigma(\{[a,b) : 2^n a, 2^n b \in \mathbb{Z}\}),$$

i.e. $\mathcal{F}_n$ is generated by intervals whose endpoints are dyadic numbers with no more than n decimal digits. Clearly $\mathcal{F}_n$ is an increasing sequence. We claim that $[0, \frac{1}{3}) \in \sigma(\bigcup_n \mathcal{F}_n) \setminus \bigcup_n \mathcal{F}_n$. Indeed, take x_n to be largest n-digit dyadic number below $\frac{1}{3}$. As dyadic numbers are dense, $x_n \to \frac{1}{3}$. Now, $[0, x_n) \in \mathcal{F}_n$, hence

$$[0, 1/3) = \cup_n [0, x_n) \in \sigma(\cup_n \mathcal{F}_n).$$

However, $[0, \frac{1}{3})$ does not belong to $\mathcal{F}_n$ for all n, as $\frac{1}{3}$ is not dyadic.

References