# Stats 310A Session 6

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## 1 The Law of iterated logarithm

In this note we prove the law of iterated logarithm, mainly following [**?** , Chapter 9]. Let $X_i$ be independent R.V.-s with mean 0 and variance 1. The central limit theorem characterizes the behavior of $S_n = X_1 + \cdots + X_n$ and states that $S_n = O_p(\sqrt{n})$. The law of iterated algorithm refines this result dramatically, precisely characterizing the scalings of the extrema of $S_n$.

**Theorem 1** (Law of iterated logarithm)**.** *We have*

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} = 1\right) = 1.$$

Equivalently, the theorem states the following: for all $\varepsilon > 0$,

$$\mathbb{P}\left(S_n \geq (1+\varepsilon)\sqrt{2n\log\log n} \text{ i.o.}\right) = 0, \tag{1}$$

$$\mathbb{P}\left(S_n \geq (1-\varepsilon)\sqrt{2n\log\log n} \text{ i.o.}\right) = 1. \tag{2}$$

Hence, showing LIL requires estimating the probability $\mathbb{P}(S_n/\sqrt{n} \geq t)$ very accurately, for $t$ on the order of $\sqrt{\log\log n}$. The following lemma presents such a result.

**Lemma 1.1.** *Let $a_n \to \infty$ and $a_n/\sqrt{n} \to 0$, then*

$$\mathbb{P}\left(\frac{S_n}{\sqrt{n}} \geq a_n\right) = \exp\left(-\frac{1}{2}a_n^2(1+\xi_n)\right),$$

*where $\xi_n \to 0$.*

We will also need a variant of Kolmogorov's maximal inequality. Let $M_n = \max_{1\leq k\leq n} S_k$ be the maximum process.

**Lemma 1.2.** *For $\alpha \geq \sqrt{2}$, we have*

$$\mathbb{P}\left(\frac{M_n}{\sqrt{n}} \geq \alpha\right) \leq 2\mathbb{P}\left(\frac{S_n}{\sqrt{n}} \geq \alpha - \sqrt{2}\right).$$

**Proof of Theorem ??** We prove the result by looking at a subsequence $S_{n_k}$ where $n_k = \theta^k$ for some carefully chosen $\theta > 1$. We bound the deviation probability carefully and use Borel-Cantelli to show that $S_{n_k}$ exceeds the desired threshold infinitely often with probability zero or one. We then show that $S_n$ has the same behavior as the subsequence.

**Proof of (??)** Fixing $\varepsilon > 0$, choose $\theta$ such that $1 < \theta^2 < 1 + \varepsilon$. Define

$$n_k = \left\lfloor \theta^k \right\rfloor, \quad x_k = \theta\sqrt{2\log\log n_k}.$$

Note that $x_k = (1 + o(1))\theta\sqrt{2\log k}$. Applying Lemmas **??**, **??**, we obtain

$$\begin{aligned}
\mathbb{P}\left(\frac{M_{n_k}}{\sqrt{n_k}} \geq x_k\right) &\leq 2\mathbb{P}\left(\frac{S_{n_k}}{\sqrt{n_k}} \geq x_k - \sqrt{2}\right) \\
&= 2\exp\left(-\frac{1}{2}(x_k - \sqrt{2})^2(1 + \xi_k)\right) \\
&= 2\exp\left(-\frac{1}{2} \cdot 2\theta^2 \log k(1 + o(1))\right) \\
&\leq \frac{2}{k^{\theta^2}},
\end{aligned}$$

the last bound holding for all large $k$. As $\theta^2 > 1$, the RHS is summable, so by Borel-Cantelli I we have

$$\mathbb{P}\left(\frac{M_{n_k}}{\sqrt{n_k}} \geq x_k \text{ i.o.}\right) = 0.$$

We now argue that $S_n \geq (1 + \varepsilon)\sqrt{2n\log\log n}$ infinitely often will happen with probability zero. Suppose it happens infinitely often, let $n$ be an index where it happens. Let $k$ be such that $n_{k-1} < n \leq n_k$. We then have

$$\begin{aligned}
\frac{M_{n_k}}{x_k\sqrt{n_k}} = \frac{M_{n_k}}{\theta\sqrt{2n_k\log\log n_k}} &\geq \frac{S_n}{\theta\sqrt{2n\log\log n}} \cdot \sqrt{\frac{2n_{k-1}\log\log n_{k-1}}{2n_k \log\log n_k}} \\
\geq \frac{1+\varepsilon}{\theta} \cdot \sqrt{\frac{2\theta^{k-1} \cdot \log(k-1)}{2\theta^k \log k}}(1 + o(1)) \\
\geq \frac{1+\varepsilon}{\theta^{3/2}}(1 + o(1)).
\end{aligned}$$

As $1 + \varepsilon > \theta^2 > \theta^{3/2}$, for sufficiently large $k$, the above quantity will be greater than one. Hence, $M_{n_k}/\sqrt{n_k} \geq x_k$ will happen infinitely often. As this has probability zero, we must have $\mathbb{P}(S_n \geq (1 + \varepsilon)\sqrt{2n\log\log n} \text{ i.o.}) = 0$, thereby showing (**??**).

**Proof of (??)** Let $\theta$ be an integer such that $3/\sqrt{\theta} < \varepsilon$ and $n_k = \theta^k$. Define

$$a_k = x_k/\sqrt{n_k - n_{k-1}} \quad \text{with } x_k = (1 - \theta^{-1})\sqrt{2n_k\log\log n_k}.$$

As $S_n$ are sums of independent R.V.-s, we can apply Lemma **??** to $S_{n_k} - S_{n_{k-1}}$ and get

$$\begin{aligned}
\mathbb{P}\left(S_{n_k} - S_{n_{k-1}} \geq x_k\right) &= \exp\left(-\frac{x_k^2}{2(n_k - n_{k-1})}(1 + \xi_k)\right) \\
&= \exp\left(-\frac{(1-\theta^{-1})^2 2\theta^k \log k}{2(1-\theta^{-1})\theta^k}(1 + o(1))\right) \\
&= \exp\left(-(1-\theta^{-1})\log k(1 + o(1))\right) \\
&\leq \frac{2}{k^{1-\theta^{-1}}},
\end{aligned}$$

the last bound holding for all large $k$. As the RHS sums up to infinity and the events are independent, by Borel-Cantelli II we get that

$$\mathbb{P}\left(S_{n_k} - S_{n_{k-1}} \geq x_k \text{ i.o.}\right) = 1.$$

We now argue that the above implies $S_{n_k} > (1-\varepsilon)\sqrt{2n_k \log\log n_k}$ happens infinitely often with probability one, thereby showing the result. Indeed, applying the established result (**??**) to $-S_{n_k}$ with $\varepsilon = 1$, we get $-S_{n_{k-1}} \leq 2\sqrt{2n_{k-1}\log\log n_{k-1}}$ for all large $k$. Combined with the above result, we get that with probability one,

$$S_{n_k} \geq x_k - 2\sqrt{2n_{k-1}\log\log n_{k-1}} \geq x_k - \frac{2}{\sqrt{\theta}}\sqrt{2n_k\log\log n_k} = \left(1 - \frac{1}{\theta} - \frac{2}{\sqrt{\theta}}\right)\sqrt{2n_k\log\log n_k}$$
$$\geq \left(1 - \frac{3}{\sqrt{\theta}}\right)\sqrt{2n_k\log\log n_k} \geq (1-\varepsilon)\sqrt{2n_k\log\log n_k}.$$

□

For completeness, we also provide the proof of Lemma **??**.

**Proof of Lemma ??** Suppose $M_n/\sqrt{n} \geq \alpha$, then either $S_n/\sqrt{n} \geq \alpha - \sqrt{2}$, or $S_n/\sqrt{n} < \alpha - \sqrt{2}$ and one of the following happens: $M_{j-1} < \alpha\sqrt{n}$ but $M_j \geq \alpha\sqrt{n}$. Defining $A_j = \{M_{j-1} < \alpha\sqrt{n} \leq M_j\}$, then

$$\mathbb{P}\left(\frac{M_n}{\sqrt{n}} \geq \alpha\right) \leq \mathbb{P}\left(\frac{S_n}{\sqrt{n}} \geq \alpha - \sqrt{2}\right) + \sum_{j=1}^{n-1}\mathbb{P}\left(A_j \cap \left\{\frac{S_n}{\sqrt{n}} \leq \alpha - \sqrt{2}\right\}\right).$$

On each of the event $A_j \cap \{\cdots\}$, we have $S_j \geq \alpha\sqrt{n}$ and $S_n \leq (\alpha - \sqrt{2})\sqrt{n}$, which implies $(S_n - S_j)/\sqrt{n} \leq -\sqrt{2}$. This event is independent of $A_j$, and $S_n - S_j$ has variance $n-j$, so we get

$$\mathbb{P}\left(A_j \cap \left\{\frac{S_n}{\sqrt{n}} \leq \alpha - \sqrt{2}\right\}\right) \leq \mathbb{P}\left(A_j \cap \left\{\frac{S_n - S_j}{\sqrt{n}} \leq -\sqrt{2}\right\}\right)$$
$$= \mathbb{P}(A_j)\mathbb{P}\left(\frac{S_n - S_j}{\sqrt{n}} \leq -\sqrt{2}\right) \leq \frac{n-j}{2n}\mathbb{P}(A_j).$$

Plugging into the preceding bound gives

$$\mathbb{P}\left(\frac{M_n}{\sqrt{n}} \geq \alpha\right) \leq \mathbb{P}\left(\frac{S_n}{\sqrt{n}} \geq \alpha - \sqrt{2}\right) + \sum_{j=1}^{n-1}\frac{n-j}{2n}\mathbb{P}(A_j) \leq \mathbb{P}\left(\frac{S_n}{\sqrt{n}} \geq \alpha - \sqrt{2}\right) + \frac{1}{2}\sum_{j=1}^{n-1}\mathbb{P}(A_j).$$

As $A_j$ are disjoint and $\bigcup A_j$ implies $\{M_n/\sqrt{n} \geq \alpha\}$, we get

$$\mathbb{P}\left(\frac{M_n}{\sqrt{n}} \geq \alpha\right) \leq \mathbb{P}\left(\frac{S_n}{\sqrt{n}} \geq \alpha - \sqrt{2}\right) + \frac{1}{2}\mathbb{P}\left(\frac{M_n}{\sqrt{n}} \geq \alpha\right),$$

from which the result follows. □