Stats 310A Session 8

December 10, 2019

In this session, we will go through some practice problems. These problems fall in the scope of Stats 310A and involves a lot of what we learned comprehensively.

Problem 1 Let $\mathcal{X}$ be a set, $\mathcal{B}$ be a countably generated σ-algebra of subsets of $\mathcal{X}$. Let $\mathcal{P}(\mathcal{X}, \mathcal{B})$ be the set of all probability measures on $(\mathcal{X}, \mathcal{B})$. Make $\mathcal{P}(\mathcal{X}, \mathcal{B})$ into a measurable space by declaring that the map $P \mapsto P(A)$ is Borel measurable for each $A \in \mathcal{B}$. Call the associated σ-algebra $\mathcal{B}^*$.

(a) Show that $\mathcal{B}^*$ is countably generated.

(b) For $\mu \in \mathcal{P}(\mathcal{X}, \mathcal{B})$, show that $\{\mu\} \in \mathcal{B}^*$.

(c) For $\mu, \nu \in \mathcal{P}(\mathcal{X}, \mathcal{B})$, let

$$\|\mu - \nu\| = \sup_{A \in \mathcal{B}} |\mu(A) - \nu(A)|.$$

Show that the map $(\mu, \nu) \mapsto \|\mu - \nu\|$ is $\mathcal{B}^* \times \mathcal{B}^*$ measurable.

Problem 2 Let $\{X_n\}_n$ be iid symmetric random variables such that

$$\lim_{y \to \infty} \frac{y^2 \Pr(|X_1| > y)}{\mathbb{E}(X_1^2; |X_1| < y)} = 0. \tag{1}$$

Show that there exists a sequence $\{b_n\}_n$ of positive constants such that

$$\frac{1}{b_n} \sum_{k=1}^{n} X_k \xrightarrow{d} \mathcal{N}(0, 1). \tag{2}$$

Problem 3 Recall that given two measures μ, ν on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, a coupling of μ and ν is any probability measure γ on $(\mathbb{R}^2, \mathcal{B}_{\mathbb{R}^2})$ such that, for any Borel set A, we have $\gamma(A \times \mathbb{R}) = \mu(A)$, $\gamma(\mathbb{R} \times A) = \nu(A)$. (In words, the one-dimensional marginals of γ are –respectively– μ and ν.) We denote by $\Gamma(\mu, \nu)$ the set of couplings of μ and ν. For $p \geq 1$, let $\mathcal{P}_p$ be the space of probability measures μ such that $\int |x|^p \, \mu(\mathrm{d}x) < \infty$. For $\mu, \nu \in \mathcal{P}_p$, their p-th Wasserstein distance is

$$W_p(\mu, \nu) = \left\{ \inf_{\gamma \in \Gamma(\mu, \nu)} \int_{\mathbb{R} \times \mathbb{R}} |x - y|^p \, \gamma(\mathrm{d}x, \mathrm{d}y) \right\}^{1/p} \tag{3}$$

1. For $\mu = \mathcal{N}(0, 1)$ and $\nu = \mathcal{N}(a, 1)$, prove that $W_2(\mu, \nu) = |a|$.
2. For $\mu = \mathcal{N}(0, 1)$ and $\nu = \mathcal{N}(0, v)$, $v > 1$, prove that $W_2(\mu, \nu) = \sqrt{v} - 1$.
3. Prove that $\Gamma(\mu, \nu)$ is uniformly tight.

4. Fix $p \geq 1$. Prove that there exists a sequence of probability measures $\{\gamma_n\}_{n\in\mathbb{R}} \subseteq \Gamma(\mu,\nu)$ amd $\gamma \in \Gamma(\mu,\nu)$ such that $\gamma_n \overset{w}{\Rightarrow} \gamma$, and

$$\lim_{n\to\infty} \int_{\mathbb{R}\times\mathbb{R}} |x-y|^p \, \gamma_n(\mathrm{d}x, \mathrm{d}y) = W_p(\mu,\nu)^p \,. \tag{4}$$

5. Prove that (for $\{\gamma_n\}_{n\in\mathbb{N}}$, γ constructed as in the previous point)

$$\liminf_{n\to\infty} \int_{\mathbb{R}\times\mathbb{R}} |x-y|^p \, \gamma_n(\mathrm{d}x, \mathrm{d}y) \geq \int_{\mathbb{R}\times\mathbb{R}} |x-y|^p \, \gamma(\mathrm{d}x, \mathrm{d}y) \,, \tag{5}$$

   and deduce that

$$W_p(\mu,\nu) = \left\{ \int_{\mathbb{R}\times\mathbb{R}} |x-y|^p \, \gamma(\mathrm{d}x, \mathrm{d}y) \right\}^{1/p} \tag{6}$$